The Pre-Marital Investment Game

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Abstract

Two sides of a finite marriage market engage in costly investment and are then matched assortatively. The purpose of the investment is solely to improve the quality of the match that trader can attain in the second stage. The paper studies the limits of equilibrium of these finite matching games as the number of traders gets large and compares them with *hedonic* solutions to the pre-marital investment problem in which levels of the costly investment are implicitly priced. It is shown that equilibria of the simultaneous investment game do not converge to the hedonic solution. A sequential version of the game has approximate equilibria in which all trader invest as they would in the hedonic solution, but other approximate equilibria with inefficiently large investment also occur in the sequential version of the game.

This paper studies the *pre-marital investment game.* There are two sides to a potentially very large market. Traders on each side of the market try to match with traders on the other side. To influence the outcome of the matching process, the traders make costly investments prior to the match in an attempt to make themselves more attractive as partners. Attention here is focussed on the case where investments are one dimensional and matching is assortative. Adding the pre-match investment phase turns this game into a prototype for a number of important applied problems.

The problem that motivated this research is primarily the family matching problem studied in (Peters and Siow 1999), and the closely related marriage problem (Cole, Malaith, and Postlewaite 2000) or (Nosaka 2002) in which potential marriage partners invest in themselves in order to attract partners. The marriage market is identical to a labour market in which workers acquire costly human capital to distinguish themselves for firms who offer different wages and working conditions (for example (Felli and Roberts 2000), (Shi 1999) or (Han 2002). An even simpler variant of this problem (Bulow and Levin 2003) is one where firms compete in wages for workers with different but exogenously given productivity.)

Perhaps a more surprising application is the hedonic pricing problem ((Rosen 1974) or (Ekeland 2003)). In this problem there are many consumers who have varying amounts of income to spend buying a commodity that can have different qualities. Firms with different cost functions produce these qualities. The 'investment' that consumers make is the amount of money they wish to offer to firms (the cost is then the foregone consumption of other goods), firms investment are the qualities of the goods that they produce. Assortative matching after these investments will pair the consumer who offers the most money with the firm who produces the highest quality, and so on. Discriminatory (double) auctions can also be viewed as pre-marital investment games. For example, in a discriminatory double auction, buyers 'investments' are the bids they make while sellers' investments are their asks. If the auction matches highest bid with lowest ask until all apparent gains to trade are realized, and buyers pay the ask of the seller with who they are matched (or sellers receive the bid of the buyer with whom they are matched) then the double auction is a pre-marital investment game.[1]

A typical method for solving the pre-marital investment problem is to extend the *hedonic* approach described above.[2] This approach is sketched below. In the marriage market, for example, each additional unit of investment in mens' human capital can be assigned an implicit return expressed in terms of the additional number of units of human capital that the man who makes this investment can expect his eventual marriage partner to have. This approach seems reasonable in large marriage markets, and is certainly a natural extension of the usual hedonic model in which characteristics are assigned money prices. Hedonic equilibria are always pareto optimal. (see (Peters and Siow 1999), (Cole, Malaith, and Postlewaite 2000), or (Han 2002)). In this interpretation, the matching market eliminates the hold up problem that

[1]The version we study here involves symmetric information, so auctions are not a very interesting example of this kind of problem.

[2]See (Peters and Siow 1999). An 'extension' to the usual approach is needed because the payoffs in the marriage problem aren't typically quasi-linear.

typically exists in bilateral matching problems when each partner's payoff depends on an investment made by the other partner.

The point of this paper is to study this process when the number of traders is finite. Here the literature is sparse. (Peters and Siow 1999) consider a very special case of the family matching problem in which there are only four families, one rich and one poor on each side. They give a relatively complete characterization of the mixed strategy equilibrium for this case. They show that investments are larger than they would be in a bilateral problem, but far smaller than efficiency would require. They illustrate the various inefficiencies associated with the mixed strategy equilibrium, including the mismatching that occurs as a consequence of the random strategies. (Felli and Roberts 2000) consider a finite version of the worker firm matching problem which involves a combination of the simultaneous and sequential game that we consider below. Firms and workers make investments simultaneously, then firms compete for workers by offering different wages once these investments have been realized. They show that this process induces efficient investments for firms *conditional on the investments made by workers*. However the investments made by workers are too low. They do not discuss what happens as the number of traders gets large, but workers' investments appear to remain inefficiently low right up to the limit.

One of the main reasons for trying to analyze Nash equilibria in more detail is to ask whether the hedonic equilibria (which are widely applied) are limits of appropriate Nash equilibria of the premarital investment game. However, the pre-marital investment game has such wide applicability that it makes sense to study properties of finite equilibria for their own sake. Two variants of the pre-marital investment problem are studied - a simultaneous move game and a sequential move game. The simultaneous move game is one in which all investments occur at the same time, well before the matching process takes place. This variant describes well the marriage problem itself, since males and females often make human capital decisions long before marriage occurs.[3] It also describes the auction application when bids and asks are submitted simultaneously. The paper begins with a very general existence theorem for Nash equilibrium in this game when there are finitely many players and symmetric information. The equilibria of this game are complex, and only a partial characterization of this equilibrium is possible.

[3]This is especially true in (Peters and Siow 1999) where families make investments in human capital on behalf of their children.

In addition, letting the number of traders become large introduces degeneracies that do not exist in the finite case. To try to make some progress the paper then proceeds by specializing the model so that payoffs are separable in own and partner's type, and so that traders on the same side of the market are all identical.

The first assumption eliminates any complementarity between own and partner's investment. The second assumption will generally tend to limit the variance of investment on each side of the market, which tends to mitigate the matching gains associated with investment. For example, if the same number of traders exists on each side of the market and traders are symmetric, then there is always a Nash equilibrium where no one invests more than they would if they were unmatched. Nonetheless the symmetric case admits a very simple (and efficient) hedonic solution when there are a continuum of traders. So it provides a very simple environment within which Nash equilibria of large games can be compared to competitive equilibria. Furthermore, if one side of the market has strictly more traders than the other side, then that side of the market always has an incentive to invest with assortative matching in order to try to ensure that they attract some partner. Since equilibrium typically involves mixed strategies, investments will be spread out on the long side, creating an incentive for the short side of the market to compete. The investments on the short side create additional incentive for the long side to invest, and so on.

The limits of Nash equilibria for the simultaneous game will not typically support efficient investment or the hedonic solution. It is in fact quite difficult to show that this must be true. The limit properties of the equilibrium allocation are difficult to identify because the equilibrium strategies on both sides become degenerate in the limit. Nonetheless, it is possible to show that limit allocations must satisfy some basic characteristics that can be shown to be inconsistent with efficient investment in general. Conceptually, however, the market failure in the simultaneous game is classic. An investor generates two kinds of externalities in the simultaneous pre-marital investment game. The first is the externality usually associated with the holdup problem. A worker, for example, who invests in human capital generates a benefit for the firm with whom he or she eventually matches. As in (Peters and Siow 1999), the existence of potential partners with different investment levels on the other side of the market partially internalizes this externality - a trader who invests more is rewarded with a better partner. However, there is a second strategic externality as well. A trader who invests more will create an incentive for traders on the other side of the market to compete to attract this better partner. This competition will make them invest more, which benefits traders on the investors own side of the market. The simultaneous pre-marital investment game does not internalize this second externality as the market gets large. Though the equilibria in the finite game are mixed, they tend as the market gets larger toward outcomes where firms make very similar investments most of the time, and match with workers who have very similar investments most of the time. These investments are inefficiently low. No firm has any incentive to raise investment because, in the simultaneous game, they simply continue to match with a worker who has about the same investment.

This brings us to consideration of the sequential version of the game. Suppose that firms inhabit the short side of the market, and make their investments before workers do. Then a firm who increases investment will anticipate that this will increase competition among workers. The return to investment is then two fold. First, increased investment will secure the firm a better worker. The investment will also increase investment by workers, so the quality of the firms partner would rise even if all his competitors matched his investment. This game form partially internalizes the strategic externalities that arise with investment.

The exact equilibria of the sequential game are again complex. Though we can show that firms investments affect the equilibrium outcome of the second stage investment game by workers continuously, firms payoffs remain discontinuous because of the jumps in payoffs that occur when ties are broken. So we show instead that when the market is large, there will exist an approximate equilibria for firms (exact equilibria for workers) such that all firms use a pure investment strategy which involves making the efficient investment. As the number of traders gets large, the payoff to both firms and workers in this approximate equilibrium converges to their payoff in the corresponding competitive (hedonic) equilibrium.

However, we also show that there are a variety of other approximate equilibrium outcomes in the sequential game when there are many traders. The payoff functions that firms face as the number of traders gets large become discontinuous in response to downward deviations in investment. When firms choose the same wage as all the others, their actual partner will be determined by randomly resolving the tie. With very high probability it won't matter much how the tie is resolved when there are many traders, the firm will get about the same quality partner. However, if the firm deviates downward in investment, it will surely match with the *worst matched worker*. As will be shown, the strategies that support equilibrium when the number of traders get large will mean that this workers' expected investment will be much worse that the investment quality of most of the others. So the deviating firms' payoff will drop discontinuously. Under reasonable conditions, this will ensure that no firm can gain by cutting investment. We show that when investment is near its efficient level, no firm has much to gain by raising investment.

1 Fundamentals

For ease of discussion this paper simply adopts the worker-firm version of the problem. The market consists of m 'firms' and n 'workers'. We use the notation M and N to refer to the sets of firms and workers respectively. Each firm has a characteristic x drawn from a closed connected interval $X \subset \mathbb{R}^+$. Each worker has a characteristic y which is again contained in a closed connected interval $Y \subset \mathbb{R}^+$. Firms and workers make investments $k \in K \subset \mathbb{R}^+$ and $h \in H \subset \mathbb{R}^+$ in physical capital and human capital respectively, where both K and H are assumed compact connected intervals.

Payoffs for firms and workers depend on their own type, on their own investment, and on the investment level of their eventual partner. The payoff of a firm is given by $v : K \times H \times X \to \mathbb{R}$, and that of a worker by $u : H \times K \times Y \to \mathbb{R}$. For convenience we write $v^0(y)$ as the maximal payoff to an unmatched firm of type x and $u^0(x)$ as the maximal payoff to an unmatched worker of type y.

It is assumed that for each (x, h) the function $v(\cdot, x, h)$ is continuous and that $v(x, k, \cdot)$ is increasing in h. Let

$$k^*(x, h) = \arg\max_k v(x, k, h)$$

and refer to k^* as the *Nash* investment level for the firm. Assume that the following single crossing condition holds: suppose $x' > x$, $k' > k$ and $h' > h$ and that $v(x, k, h) = v(x, k', h')$, then $v(x', k', h') > v(x', k, h)$. The single crossing condition implies that if $k^*(x, h)$ is unique and on the interior of K then it is strictly increasing in x.

Assume the payoff functions are supermodular in investments in the sense that for any x, and for any (k, h) and (k', h'),

$$v(x, k, h) + v(x, k', h') \leq v(x, \max[k, k'], \max[h, h']) + v(x, \min[k, k'], \min[h, h'])$$

One implication of this assumption is that the Nash investment is non-decreasing in h.[4] Make similar assumptions for workers and let $h^*(y,k)$ be the Nash investment level for the worker.

It is assumed throughout that there are at least as many workers as firms. A worker who is unmatched is assumed to receive a payoff $u^0(h,x)$. Assume workers always strictly prefer to be matched with some firm.

The game form is straightforward whether the game is simultaneous or sequential. In the simultaneous version, workers and firms simultaneously choose investments and are then assortatively matched based on these investments.[5] In the sequential game, it is assumed that firms invest first, workers observe these investments before making their own investments. Then assortative matching occurs.

Specifically, taking the investments of the workers and firms to be given as $\{k_i\}_{i=1,n}$ and $\{h_j\}_{j=1,m}$ the one to one matching function π from M into N satisfies the assortative matching property if $k_i > k_{i'} \Rightarrow h_{\pi(i)} \geq h_{\pi(i')}$.

The matching function π depends on the arrays of investments made by the different traders. We assume additionally that π satisfies the following pair of *matching regularity conditions*:

1. let (k_i, k_{-i}) and (k_i', k_{-i}) be two arrays of investments for the sellers that differ only in their i^{th} component. Suppose that $h_{\pi(i:(k_i,k_{-i}))} = h_{\pi\left(i:k_i',k_{-i}\right)}$. Then the same is true for every j, i.e., for each j $\pi(j:(k_i,k_{-i})) = \pi(j:(k_i',k_{-i}))$. Similarly for workers.

[4]Fix x, and $h < h'$. Let k be Nash against h and k' be Nash against $h'$and suppose that $k' < k$. Then by supermodularity

$$v(x,k,h) + v(x,k',h') \leq$$

$$v(x,k',h) + v(x,k,h')$$

Since v is assumed to be single peaked, $v(x,k',h) < v(x,k,h)$. Then supermodularity requires that

$$v(x,k,h') > v(x,k',h')$$

which is a contradiction.

[5]Assortative matching is simply assumed. It is possible to add the matching process to the extensive form of the game in such a way that matching occurs as an equilibrium outcome. Since this kind of argument is well know from (Roth and Sotomayor 1990) it seems reasonable to take it for granted.

2. Let ρ be any matching function that satisfies the assortative matching condition. Then

$$\sum_{i=1}^{n} v\left(x_i, k_i, h_{\pi(i)}\right) + u\left(y_{\pi(i)}, h_{\pi(i)}, k_i\right) \geq$$

$$\sum_{i=1}^{n} v\left(x_i, k_i, h_{\rho(i)}\right) + u\left(y_{\rho(i)}, h_{\rho(i)}, k_i\right)$$

The first condition simplifies the proof of the main existence theorem slightly, but is not actually necessary. It is plausible enough that is seems worthwhile to take advantage of it. It specifies that if a change in some trader's investment does not affect the investment level of his partner, then it does not affect the investment level of any trader's partner. The second condition is the important one for the existence theorem. It says that ties are resolved in a way that maximizes the sum of the surplus generated by the match.

Given any array of investments, the matching function pairs workers and firms and determines which workers are unemployed. In particular this pairing determines the payoff of every firm and worker for any array of investment. The simultaneous pre-marital investment game is just the normal form game determined by this payoff function. An equilibrium for the game is just a Nash equilibrium for this game. The sequential move game is the two stage game with these same payoffs. An equilibrium for the sequential game is a subgame perfect equilibrium for the two stage game.

2 Hedonic Equilibrium

This section digresses briefly to describe the hedonic approach to the pre-marital investment game as describe by (Peters and Siow 1999). As it is straightforward to show that hedonic equilibria are efficient and efficient allocations coincide with hedonic equilibria in the specialized environment considered below, the problem discussed in this paper is whether or not equilibrium of the simultaneous or sequential move game are close to hedonic equilibria when the number of traders is large.

In this case let F and G be the *measures* of firms and workers (on the spaces X and Y) in the economy respectively. Let $Z = K \times H$. Let

$\left(k^f(\cdot), h^f(\cdot)\right) : X \to Z$ and $(k^w(\cdot), h^w(\cdot)) : Y \to Z$ be measurable allocations. These allocations are feasible if the measures they induce on Z are the same (which means that some workers must remain unmatched). The allocations are pareto optimal if there does not exist an alternative feasible allocation for which all workers and firms are at least as well off, with some worker or firm being made strictly better off.

Let $w(h)$ be a *hedonic* price functional. The interpretation is that $w(h)$ is the amount that a firm needs to pay to 'buy' a worker whose human capital is h. The same firm finances this transaction by 'selling' its capital at unit price 1. It is important to note that this price functional is *not* linear. An allocation $\left\{\left(k^f(\cdot), h^f(\cdot)\right), (k^w(\cdot), h^w(\cdot))\right\}$ is a hedonic pricing equilibrium if for all x and y, $v\left(k^f(x), h^f(x), x\right) \geq v^0(x)$; $u(h^w(y), k^w(y), y) = u^0(y)$ and if there exists a hedonic price functional $w(\cdot)$ such that

$$v\left(k^f(x), h^f(x), x\right) \geq v(k', h', x)$$

for all (k', h') such that

$$k' - w(h') = 0$$

and

$$u(h^w(y), k^w(y), y) \geq u(h', k', y)$$

for all (h', k') such that

$$w(h') - k' = 0$$

Generally $w(\cdot)$ is not a money price. In (Peters and Siow 1999) $w(h)$ it is capital investment of the firm that will hire the worker who invests h. There is one case in which this reduces exactly to the hedonic pricing problem as discussed by (Rosen 1974) or (Ekeland 2003). To see this special case, suppose that

$$v(k, h, x) = \tilde{v}(h, x) - k$$

and that

$$u(h, k, y) = k - \tilde{u}(h, y)$$

Think of k as a money payment that firms choose to make to workers who are essentially producing products of different qualities h with cost functions that depend on their characteristic. This is the standard hedonic pricing problem.[6][7]

This hedonic pricing problem is, in turn, an extension of the *directed matching problem* when workers have observably different qualities. In this interpretation firms produce aggregate revenues $\tilde{v}(h, x)$ which depend on their own intrinsic technology x and the human capital h of the worker they are able to hire. The firms post wages w before matching occurs. Each worker then makes an application to (one and only one) firm in full knowledge of the wages each firm offers or alternatively each firm bids for the services of a worker. Examples of models of this sort are (Shi 1999) and more recently (Bulow and Levin 2003). These models are special cases of the general hedonic pricing problem since the human capital investments of workers are assumed exogenous.

We can illustrate how to construct this hedonic equilibrium.[8] The argument follows (Peters and Siow 1999). Let $\theta(x)$ be the function that matches workers and firms assortatively, with $\theta(\underline{x})$ chosen such that $F(X) = G([\theta(\underline{x}), \overline{y}])$ (here $\underline{x}$, $\underline{y}$, $\overline{x}$, and $\overline{y}$ are the lowest and highest types in the sets X and Y).

For any (x, k) define

$$h^e(x, k) = \left\{ h' : \frac{u_h(h', \theta(x), k)}{u_k(h', \theta(x), k)} = \frac{v_h(k, x, h')}{v_k(k, x, h')} \right\} \tag{1}$$

Fixing a firm's type *and investment*, this function gives the human capital investment that is bilaterally efficient when this firm is (assortatively) matched with a worker of type $\theta(x)$.

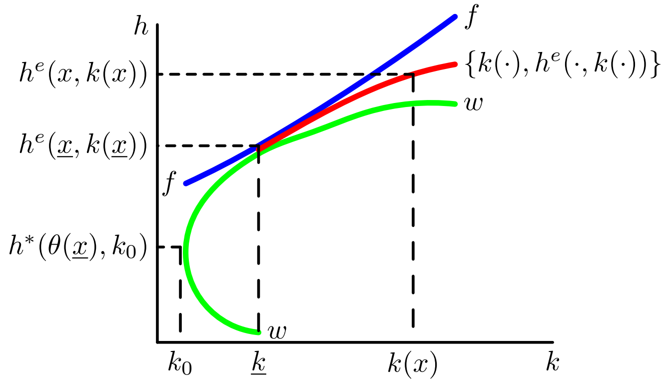
Choose $\underline{k}$ to satisfy

$$u(h^e(\underline{x}, k'), k', \theta(\underline{x})) = u^0(\theta(\underline{x}))$$

[6]More generally, the hedonic pricing problem allows the characteristic that the firm chooses, h in this case, to be drawn from a much larger dimensional space. However, the hedonic pricing problem doesn't allow the dimension of the consumer's characteristic to be higher than 1 - the consumer always pays money for the different qualities of good.

[7]The 'price' of capital in this story is normalized to 1. This wouldn't work if the characteristic chosen by firms was of high dimension. The hedonic pricing equilibrium can still be defined, but the pricing function would have to a more general non-linear functional of investment.

[8]Note that there is no money in this problem, so calling this a hedonic *pricing* equilibria is misleading.



so that the worker who is assortatively matched with this firm is just indifferent about matching.

For any non-decreasing function $k(x)$, the relation $x \rightarrow (k(x), h^e(x, k(x)))$ with $k(\underline{x}) = \underline{k}$ defines a non-decreasing locus in the space $\mathbb{R}^2$ of capital investments because of the single crossing condition. Call this locus $h^*(k)$. The slope of this locus is derived in a straightforward manner from $k(x)$ according to

$$\frac{dh^*(k)}{dk} = \frac{dh^e(x, k(x))}{dk(x)} = \frac{h_x^e(x, k(x))\, dx + h_k^e(x, k(x))\, k'(x)\, dx}{k'(x)\, dx}$$

At each point along this locus the indifference curves of a firm of type x and a worker of type $\theta(x)$ are tangent. If we pick h^* so that these indifference curves are also tangent to h^*, in other words

$$\frac{h_x^e(x, k(x)) + h_k^e(x, k(x))\, k'(x)}{k'(x)} = \frac{v_k[k(x), x, h(x, k(x))]}{v_h[k(x), x, h(x, k(x))]} \tag{2}$$

then no trader will be able to do better than the tangency point if he or she believes that they can select any point along h^*. (Peters and Siow 1999) interpreted h^* as the market return to investment for firms, which meant that firms who thought they could match with a partner of type $h^*(k)$ by investing k would uniquely choose the tangency point. The conditions for which (2) has a unique solution are straightforward - see (Peters and Siow 1999).

The logic behind this construction is depicted in Figure 2.

There is a family of curves like ff representing iso-profit curves for firms. A family of curves like ww representing iso-payoff lines for workers. The shapes of these curves depend on the types of the firms and workers who own them. Some workers won't be able to match with firms. In an efficient solution, these workers should be the ones with the lowest type (for example, highest cost of acquiring eduction), since their incentive to invest is weakest. The measure of this set of workers will be $G(Y)-F(X)$. Each of the workers in this group should anticipate this and choose their Nash investment as if they were matching with a firm whose investment is k_0.

The 'worst' worker who matches will have type $\theta(\underline{x})$. This worker should be just indifferent between remaining unmatched, and matching with the worst firm whose type is $\underline{x}$. This is what happens in the diagram. The line labelled ww is an iso utility curve for a worker of type $\theta(\underline{x})$ and this line passes through the point $(k_0, h^*(\theta(\underline{x}), k_0))$.

To see how this is related to the standard hedonic pricing problem, simply compute $w(h)$ by finding the vector that is perpendicular to the tangent to $h^*(k)$ at h normalized so that its first component is 1. This vector describe the hedonic price function that supports this rational expectations outcome.

To see why it makes sense to consider the Nash equilibria of finite versions of this game, consider the investment undertaken by a firm with the lowest type $\underline{x}$. This firm can't do worse than the indifference curve ff that it attains when it makes the efficient investment $\underline{k}$ and matches with a worker of type $\theta(\underline{x})$ who has invested $h^e(\underline{x}, k(\underline{x}))$. However it could consider cutting its investment below $\underline{k}$.

In the hedonic market interpretation, cutting investment is the same as visiting an inactive market. Of course, the hedonic pricing equilibrium requires that a price be defined in this inactive market, and this price has to be such that the deviation by firm $\underline{x}$ is unprofitable. Any increasing hedonic price function whose slope at $(\underline{k}, h^*(\underline{k}))$ is equal to the slope of firm $\underline{x}$'s indifference curve at this point will make the deviation unprofitable for a firm of type $\underline{x}$. However in the matching problem following the investments, the worker of type $\theta(\underline{x})$ would have no better alternative than to continue to match with the firm even if it lowered its investment. If the firm understood this, the entire equilibrium would un-ravel from below.

This motivate the question addressed in this paper - does the hedonic pricing equilibrium approximate the behavior of traders in some reasonable non-cooperative game.

3 Equilibria with finite numbers of Firms and

Workers in the Simultaneous Game

The simultaneous pre-marital investment game typically won't have pure strategy equilibria.[9] The payoff functions are discontinuous. For example, if two firms have the same investment, then one of them can strictly improve the expected quality of his or her partner by investing just slightly more than some other firm. To begin the discussion of the matching process we in the finite case, we provide a fairly general existence theorem. The discontinuities in the game mean that standard theorems won't apply. Furthermore, the game has an unusual structure. Firms, for example, compete against other firms for partners. This is wholly standard, like a Bertrand pricing game, or an auction. Unlike standard games, the gains to winning the competition cannot be specified exogenously - they depend on the investment decisions of workers, which are endogenous.

Let $\mu = \{\mu_1, \dots \mu_n, \mu_{n+1}, \dots \mu_{n+m}\}$ be the vector describing the mixed strategies of the firms, then the workers. The arguments that follow are completely symmetric for workers and firms, so focus on firms' payoffs.

For any single firm, write the mixed strategies of the other firms and workers as μ_{-i}. Let $R_j(k_i, k_{-i}, h)$ be the probability that firm i is matched with the worker who has the $j + (n-m)^{th}$ best human capital investment given the matching function[10] given the investments of the other firms and workers. The payoff of firm i can be written as

$$V_i(x_i, k_i, \mu_{-i}) \equiv$$

$$\int \cdots \int \sum_{j=1}^{m} R_j(k_i, k_{-i}, h)\, v\left(x_i, k_i, h_{(j+n-m)}\right) d\mu_{-i} \qquad (3)$$

where $h_{(j+n-m)}$ means the $j + n - m^{th}$ order statistic among the n human capital investments of the workers.

Theorem 1 *If the matching function satisfies the matching regularity properties, then the investment game described by (3) (and the corresponding formula for workers) has at least one mixed strategy Nash equilibrium.*

[9] See the example in (Peters and Siow 1999) which illustrates why.

[10] There are $n - m$ more workers than firms. The lowest human capital investments will be made by workers who do not expect to match.

The proof is included in the appendix. Little can be said about the nature of equilibrium in the general case, except that firms will invest more on average than they will when they are alone.

3.1 Separable Case

This section adopts the assumption that traders' payoffs are separable in their own and their partner's investment.[11] Furthermore, with separability, no essential loss in generality is involved by assuming that the utility functions can be written in the quasi-linear form $v(x_i, k) + h$ and $u(y_i, h) + k$.[12]

The quasi linear formulation provides two special properties. First, every matching function that satisfies the assortative matching condition generates the same summed surpluses. So when there are ties among firms for example, we can treat the firms symmetrically and give them each an equal chance of matching with the better worker. In the non-separable case, the firm with the higher endowment typically does better in the case of a tie. A second desirable property, as will be apparent shortly, is that the separable case lends itself to the interpretation that the expected quality of the firms worker is a type of return function for investment.

The monotonicity assumption is that $u(x_i, k)$ has a unique maximizer (which obviously doesn't depend on h) which is strictly increasing in the seller's type x_i. Similar assumptions are imposed on u. The k that maximizes $u(x_i, k)$ is the investment that the firm will make when his or her partner's quality is fixed, no matter what that quality happens to be. Thus we can speak of the firms *Nash* investment level $k^*(x_i) = \arg\max u(x_i, k)$.

The human capital that a firm is able to attract depends on its own investment, on the realizations of the other firms' investments, and on the realized human capital investments of workers. The last observation creates most of the difficulties associated with characterizing equilibrium. To establish properties of the equilibrium strategies for firms, it is necessary to know something about the equilibrium strategies of workers. These in turn depend on establishing properties of firms strategies. This circularity is very difficult

[11] This assumption was used in (Peters and Siow 1999), for example.

[12] Observe that this quasi-linearity assumption is different from the one used in a pure hedonic pricing equilibrium. For example if firms offering different qualities match with consumers who offer different monetary payoffs, both consumers and firms payoffs are assumed quasi-linear in money to get the standard hedonic pricing result. Here quasi-linearity applies to the partner's characteristic.

to resolve in the general case. To proceed we begin by examining some of the properties of equilibrium that do hold generally. To get a convergence result, however, attention if focussed on the very special case where all workers are identical and all firms are identical (though there are different measures of both).

Fix the strategies μ_{-ij} of the workers and the firms other than i and j. In all of the arguments that follow, we consider finite markets where the number of workers is τ times the number of firms, i.e., $n = \tau m$. So we can index each market by the number of firms m who are active. The expected values of the order statistics of the investments of workers are given by some non-decreasing finite sequence $\left\{\overline{h}_{(1)}, \overline{h}_{(2)}, \ldots \overline{h}_{(n)}\right\}$. Let $\overline{H}_n$ be the corresponding distribution generated by these order statistics. For any $x \in [0,1]$, define the function

$$\overline{H}_m^{-1}(x) = \begin{cases} \overline{h}_{(l)} : \max\{l : n - l \geq m(1-x)\} & \text{if } \frac{1}{m} \leq x \leq 1 \\ \overline{h}_{(1)} & \text{if } x < \frac{1}{m} \end{cases}$$

Suppose a firm makes the $\frac{x}{m}^{th}$ lowest investment among firms. Then there will be $m(1-x)$ firms who have a higher investment. The function $\overline{H}_m$ says that in this case the firm will match assortatively with a worker whose human capital investment ranks l^{th} where l is chosen so that there are $n-1 = m(1-x)$ workers whose human capital investment is higher. This worker will on average have human capital investment $\overline{h}_{(l)}$. The inequalities and the restriction when $x < \frac{1}{m}$ become less 'important' when m and n are large and are needed only so the function can be defined on all of $[0,1]$.

The return associated with any investment k by a firm depends on the rank of that investment relative to the investments of the other firms as well as on the distribution of investments of the workers. This rank is random both because the other investments are random, and because ties are broken randomly.

Now choose any pair of firms, say the i^{th} and j^{th}. Given an array of investments $\tilde{k}_{-ij}$ by the firms other than i and j, the rank attained by firm i conditional on j investing *less* than i is unique unless one or more of the firms other than i and j choose exactly the same investment as i does. In that special case i's rank is uniformly distributed over a number of adjacent ranks, where the number is equal to the number of firms who have the same investment as i, plus one.[13] Think of i's rank when he invests k as a random

[13]This assumes that whenever there is a tie, the outcome is decided in favour of each of

variable x that is conditional on the investments $\tilde{k}_{-ij}$ of the firms other than i and j. Then for any realization $\tilde{k}_{-ij}$ and $\tilde{h}$ of firms' and workers' investments, we can write the expected payoff associated with an investment of k conditional on i beating j as $\mathbb{E}_{x|k}\overline{H}^{-1}(x)$.

Each mixed strategy μ_i is right continuous. Let $\mu_i^-(k) = \lim_{k' \uparrow k} \mu_i(k)$. If $\mu_i(k) - \mu_i^-(k) > 0$ then μ_i has an atom at k. Given firm j's mixed strategy μ_j, the expected quality of the worker with whom firm i is matched is given by

$$\overline{r}^n(k_i) = \mathbb{E}_{\tilde{k},\tilde{h}}\mathbb{E}_{x|k}\cdot$$

$$\mu_j^-(k_i)\overline{H}_m^{-1}(x) + (1-\mu_j(k_i))\overline{H}_m^{-1}\left(x - \frac{1}{n}\right) + \left(\mu_j(k_i) - \mu_j^-(k_i)\right)\frac{\overline{H}_m^{-1}(x) + \overline{H}_m^{-1}\left(x - \frac{1}{n}\right)}{2}$$

Observe that in a market with n firms, x cannot take a value smaller than $1/n$ with positive probability, so that we don't need to worry about the value of $\overline{H}_m^{-1}$ when x is negative. The functions

$$\overline{r}_{ij}^n(k) = \mathbb{E}_{\tilde{k},\tilde{h}}\mathbb{E}_{x|k}\overline{H}_m^{-1}(x) \tag{4}$$

and

$$\underline{r}_{ij}^n(k) = \mathbb{E}_{\tilde{k},\tilde{h}}\mathbb{E}_{x|k}\overline{H}_m^{-1}\left(x - \frac{1}{n}\right) \tag{5}$$

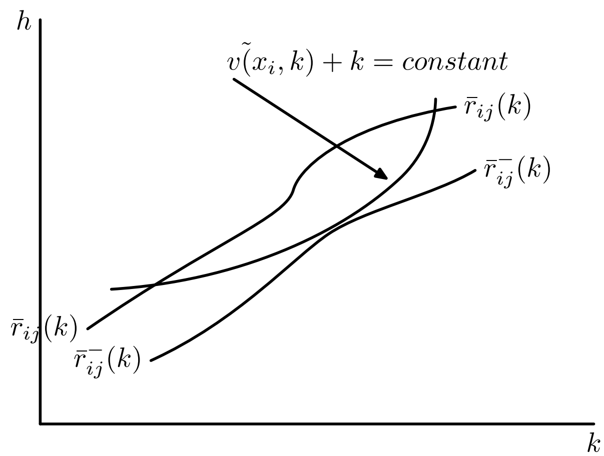
are drawn in Figure 3. It should be apparent that $\overline{r}_{ij}(k) \geq \underline{r}_{ij}(k)$. Observe as well that since we have assumed ties are broken in a way that is independent of the types of the sellers involved, the functions $\overline{r}_{ij}$ and $\underline{r}_{ij}$ are the same for both firms i and j. An iso payoff curve is drawn for firm i in the picture. Much of the logic of the convergence proof that follows is driven by the observation that the support of firm i's mixed strategy must be contained in the line segment between the point where this isopayoff curve is tangent to $\underline{r}_{ij}(k)$ and the point where it cuts through $\overline{r}_{ij}(k)$

Our first lemma provides a simple, yet intuitive relationship between the mixed strategies used by sellers of different types

Lemma 2 *Let k be a point in the convex hull of the support of the mixed strategy of firm i. Then*

---

the tied firms with equal probability.



1. $k \geq k_i^*$

2. if j *is any firm such that* $x_j > x_i$ *then* $\mu_j(k) \leq \mu_i(k)$ *for all* $k \geq k_i^*$.

Firms never invest less than their Nash investment and mixed strategies have a monotonicity property. Generally, equilibrium strategies will have atoms (see the example in (Peters and Siow 1999)). However, they will have joint atoms only under unusual circumstances.

Lemma 3 *In any equilibrium, if any pair of firms* i *and* j *has a joint atom at some investment level* k_0 *then* $\mathbb{E}_{x|k}\overline{H}_n^{-1}\left(x, \tilde{h}\right) = \mathbb{E}_{x|k}\overline{H}_n^{-1}\left(x - \frac{1}{n}, \tilde{h}\right)$ *with probability* 1.

Proof. This is a simple consequence of the discontinuities in the payoff function. If this were false, then either of the firms could discretely raise the expected quality of its partner with strictly positive probability by increasing its investment infinitesimally. ■

The argument behind Lemma 3 is completely standard, except for one interesting problem. It can't be show, as in the Bertrand game for example, that there won't be any atoms at all. The reason is that though increasing investment will raise the firms rank, this may not increase its payoff if the higher ranked workers are expected to have the same investment. This latter bit of information is endogenous.

At this point we show a result that is reminiscent of a result proved in (Allen and Hellwig 1986). Let μ_{i_n} denote the equilibrium mixed strategy of worker i.

Theorem 4 *For every n, the nash investment $k^* (\underline{x})$ must be in the support of the equilibrium strategy of the firm with the lowest type.*

Proof. Let $\overline{k}_1$ be the minimum point in the support of firm 1's equilibrium strategy. Since firm 1 has the lowest type, by Lemma 2, no firm invests less that $\overline{k}_1$. Furthermore, one of two things must be true - either every other firm invests strictly more than $\overline{k}_1$ with probability 1, or $h_{(2)} = h_{(1)}$ with probability 1. This latter observation follows because if any firm plays $\overline{k}_1$ with strictly positive probability, firm 1 could profitably deviate by raising investment slightly to break ties unless $h_{(2)} = h_{(1)}$ almost surely. So if firm 1 plays $\overline{k}_1$ he must be matched with the worker who has the lowest capital investment almost surely. However, he would continue to be so matched if he were to cut investment slightly. So a profitable deviation exists unless $\overline{k}_1$ $= k_1^*$. ■

An immediate corollary is that the Nash investment $h^* (y)$ must be in the support of the equilibrium strategy for each worker whose type is $\theta (\underline{x})$ or below.

4 Identical Workers and Firms

Now assume that traders on the same side of the market are identical. Suppose that $m = \tau n$ and $\tau > 1$. To satisfy a regularity condition we add the assumption that workers can have one of two possible types - *regular* and *poor*. A regular worker whose investment is h and who matches with a firm whose investment is k has payoff

$$u (h, k) = u (h) + k$$

A regular worker who doesn't match has payoff $u (h)$. All firms whose investment is k and who match with a worker whose human capital investment is h get payoff

$$v (k, h) = v (k) + h$$

Any firm who doesn't match has payoff $v (k)$.

Poor workers receive the same payoff no matter who they match with. For simplicity assume that they simply make some fixed investment that is *strictly* less than h^*. Each worker is a regular worker with probability $1 - \lambda$ where λ should be though of as a small positive number.

For future reference, define (h^{**}, k^{**}) as the solution to

$$u(h^{**}) + k^{**} = u(h^{*})$$

and

$$v(k^{**}) + h^{**} = v(k^{*})$$

If a firm or worker invests more than k^{**} or h^{**} respectively, then they or the trader they match with must get a lower payoff than they could get by playing Nash and staying out of the matching process. Attention is restricted henceforth to equilibria where strategies are restricted to the intervals $[h^{*}, h^{**}]$ and $[k^{*}, k^{**}]$.

The following regularity condition will be required below: Let $\mathcal{U}$ be the family of indifference curves for workers defined on the interval $[h^{*}, h^{**}]$ and defined by $\{(h,k) : h \in \{h^{*}, h^{**}\}; \nu \geq u(h^{*}); u(h) + k = \nu\}$. The family $\mathcal{U}$ is assumed to be equi-continuous.

To 'overload' some previous notation, let F represent the common mixed investment strategy used by all firms (where $F(k)$ is the probability with which the rational firm makes an investment less than or equal to k) in the simultaneous move game. Let $\tilde{G}$ describe the common mixed strategy for regular workers in both games. (These represented measure of workers and firms in the section on hedonic pricing, but this should cause no confusion.) With this formulation, the ex ante probability with which a worker invests h or less for $h \geq h^{*}$ is $G(h) = \lambda + (1-\lambda)\tilde{G}(h)$. The discussion that follows will be focussed on G instead of $\tilde{G}$. Observe that with this interpretation, G always has an atom of size λ at the investment level $\underline{h}$ chosen by poor workers.

For any vector $k \in \mathbb{R}^n$ let $O(k)$ be the corresponding vector of order statistics associated with k. Observe that $O(k)$ is a continuous function of its arguments. We use the function $O(\cdot)$ for vectors of various dimensions, the dimension of $O(\cdot)$ should then be clear from the context.

Rather than referring to Theorem 1, we deal with the existence issue in a slightly different way.

Lemma 5 *If symmetric equilibrium strategies G and F exist then they contain no atoms on the intervals $[h^{*}, h^{**}]$ or $[k^{*}, k^{**}]$.*

Proof. In the symmetric case if all firms play an atom at some investment $k \geq k^{*}$ then there is a strictly positive probability that all rational firms will play k. Further, there is a strictly positive probability that all firms will be rational. In that event the firm will be matched with a partner whose investment is equally likely to be anything from the expectation of the lowest to the expectation of the highest order statistic among investments of workers. Since the ex ante distribution of investments by firms is monotonic, these expectations are distinct. So a firm can strictly increase the quality of its partner by increasing its investment slightly in this event. In every other event, the continuity of the payoff function in own investment ensures that the cost of raising investment can be made arbitrarily small. ■

Lemma 6 *If symmetric equilibrium strategies G and F exist, then their supports (apart from the atom at $\underline{h}$ for workers) are convex and contain h^* and k^*.*

Proof. The last part follows from Theorem 4. The argument for the first part is almost identical - if there is a gap in the support, any trader investing at the upper end of the gap can increase his payoff by lowering his investment since this won't change his probability of matching. ■

The fact that there are no atoms in the supports of the firms and regular workers equilibrium strategies makes it possible to simplify the payoffs at little bit since we no longer need to worry about ties. Given F, the vector of expected values of the order statistics of firms investments in the simultaneous move game is

$$\mathbb{E}_F O(k) \equiv$$

$$\int \cdots \int O(k_1, \ldots k_n)\, dF(k_1) \ldots dF(k_n)$$

Similarly write $\mathbb{E}_G O(h) \in \mathbb{R}^m$ as the expectation of the order statistics among the investments of workers. These functions are both continuous in the sense that if $F^n \to F$ weakly, then $\mathbb{E}_{F^n} O(k) \to \mathbb{E}_F O(k)$ by the continuity of O.

Since ties occur with zero probability, a worker who invests h will match with a partner whose expected investment is

$$\sum_{t=m-n}^{m-1} \binom{m-1}{t} G^t(h)\,(1-G(h))^{m-1-t}\, \mathbb{E}_F O_{t-(m-n-1)}(k)$$

The similar expression for firms is

$$\sum_{t=0}^{n-1} \left(\begin{array}{c} n-1 \\ t \end{array} \right) F^t(k)(1-F(k))^{n-1-t} \mathbb{E}_G O_{t+(m-n)}(h)$$

The indices of the summation are different because firms always match, while workers may not.

To ease notation slightly, let $\overline{h}_t = \mathbb{E}_G O_t(h)$ and let

$$\overline{k}_t = \begin{cases} \mathbb{E}_F O_{t-(m-n-1)}(k) & \text{if } 0 < t \leq n \\ 0 & -(m-n+1) \leq t \leq 0 \end{cases}$$

By Lemma 6 it must be that in any symmetric equilibrium

$$v(k) + \sum_{t=0}^{n-1} \left(\begin{array}{c} n-1 \\ t \end{array} \right) F^t(k)(1-F(k))^{n-1-t} \overline{h}_{t+(m-n)} \tag{6}$$

is constant for every k in the support of F and

$$u(h) + \sum_{t=m-n}^{m-1} \left(\begin{array}{c} m-1 \\ t \end{array} \right) G^t(h)(1-G(h))^{m-1-t} \overline{k}_{t-(m-n-1)} \tag{7}$$

is constant for each h in the support of G.

In particular, consider an equilibrium in which regular buyers and sellers both use mixed strategies whose support includes h^* and k^*. Then the equilibrium strategies must support constant payoffs on the intervals $\left[k^*, \overline{k}\right]$ and $\left[h^*, \overline{h}\right]$ respectively. The value of these payoffs can't be stated precisely except that firms payoff must strictly exceed $v(k^*)$ because there are more workers than firms and all workers make strictly positive investment, and a regular workers' payoff must strictly exceed $u(h^*)$ because when the number of traders is finite, there is a strictly positive probability that all other workers will be poor workers.

If the payoffs are constant, the derivative of payoffs is uniformly zero on these intervals. Focus on workers' payoffs. This means that

$$u'(h) +$$

$$\frac{dG(h)}{dh} \sum_{t=0}^{m-1} \left(\begin{array}{c} m-1 \\ t \end{array} \right) G^{t-1}(h)(1-G(h))^{m-2-t} \overline{k}_{t-(m-n-1)} (t - G(h)(m-1))$$

$$= 0$$

and similarly for firms payoffs.

This notation makes it possible to write the ordinary differential equation that characterizes the equilibrium strategy for workers (for example) as

$$\frac{dG(h)}{dh} = \frac{-u'(h)}{\sum_{t=0}^{m-1} \binom{m-1}{t} G^{t-1}(h)(1-G(h))^{m-2-t}\overline{k}_{t-(m-n-1)}(t-G(h)(m-1))} \tag{8}$$

A solution to (8) that has a strictly positive density on some sub-interval $[h^*, h'] \subset [h^*, h^{**}]$ is called a *symmetric equilibrium strategy* for workers.

Lemma 7 *For each array $\overline{k}_t \in [k^*, k^{**}]^n$ there is a unique symmetric equilibrium strategy G for workers.*

A similar theorem applies for firms, however, it relies on the possibility that the worker is poor and simply invests h_0.

Corollary 8 *The solution G to 8 exists and varies continuously (in the sup norm) with the vector of order statistics $\overline{k}$.*

Proof. Standard theorem - for example (Kreider, Kuller, and Ostberg 1968) Theorem 9-12 p 393. ■

This leads to the main theorem in this section.

Theorem 9 *A symmetric Nash equilibrium for the premarital investment game exists.*

We give the proof explicitly since it is not hard, and it illustrates the main complication associated with analyzing the pre-marital investment game.

Proof. Let $\overline{k}$ and $\overline{h}$ be arbitrary vectors of order statistics in $[k^*, k^{**}]^n$ and $[h^*, h^{**}]^n$ respectively. Let $G_{\overline{k}}$ and $F_{\overline{h}}$ be the unique symmetric equilibrium strategies associated with $\overline{k}$ and $\overline{h}$ as given by Lemma 7. Let $\overline{k}' = \mathbb{E}_{F_{\overline{h}}}O(k)$ and $\overline{h}' = \mathbb{E}_{G_{\overline{k}}}O(h)$. The transformation $(\overline{h}, \overline{k}) \to (\overline{h}', \overline{k}')$ maps the compact convex set $[h^*, h^{**}]^n \times [k^*, k^{**}]^n$ into itself, since the supports of $G_{\overline{k}}$ and $F_{\overline{h}}$ are both contained in $[h^*, h^{**}]^n$ and $[k^*, k^{**}]^n$. This transformation is continuous as the composition of two continuous operations. The solutions $F_{\overline{h}}$ and $G_{\overline{k}}$ are both sup norm continuous by Corollary 8. As the order operators O are continuous, and convergence in the sup norm implies weak convergence, the expectation operators are both continuous in F and G. Hence by Kakutani's fixed point theorem, there is a fixed point supported by symmetric mixed strategies F^* and G^* say. These mixed strategies include h^* and k^* in their supports. No firm or worker will invest less than h^* or k^* and by construction, no firm or worker can gain by playing any pure strategy in the supports of G^* or H^*. So they will constitute a Nash equilibrium provided no worker or firm can profit by deviating to an investment that lies strictly above the supports of these mixtures. Since such a deviation will lead to the same match the deviator would have made by playing at the top of the support of G^* or H^*, any such deviation will be unprofitable. Hence the mixtures that support the fixed point constitute a Nash equilibrium for the game. ■

4.0.1 Inefficiency of Nash Equilibria in the limit

The objective in this section is to show why symmetric Nash equilibria will typically be inefficient even when the number of traders is very large. The equilibrium mixed strategies become degenerate in the limit, so this can't really be done by example. Instead, we illustrate three properties that the equilibrium payoffs of worker and firms must satisfy in the limit, and show that these are inconsistent with the efficient outcome in a very broad class of economies. This involves some characterization of the equilibrium of the workers' game near the limit. Since this characterization is also used in the next section, and since it constitutes the main contribution in the paper, we will give many of the proofs explicitly.

The results that follow make heavy use of the following argument - let $\hat{G}_n$ and $\hat{F}_n$ denote the empirical distribution functions of investment associated with a particular realization of play in which actual investments are independently drawn from the *equilibrium* distributions G_n and F_n. These empirical distribution functions are random variables. Any sequence of empirical distribution functions generated by drawing from G_n and F_n as the number of traders increases is a random sequence. Then (Shorak and Wellner 1986) prove that

Lemma 10 $\left\|\hat{G}_n(\cdot) - G_n(\cdot)\right\|$ *(or* $\left\|\hat{F}_n(\cdot) - F_n(\cdot)\right\|$*) converges almost surely to zero with* n*, where* $\|\cdot\|$ *is the sup norm for functions on* $[h^*, h^{**}]$.

Now consider a firm who plays his or her nash investment. Since the Nash investment is in the support of firms equilibrium strategy for all n by Lemma 6, his or her payoff must coincide with firms' equilibrium payoff. Since this firm will have the lowest investment among firms with probability 1 in equilibrium by Lemma 5, this firm will match with the worker who has the $m - n + 1^{st}$ lowest investment. The expectation of this investment is h_{1n} such that $G_n(h_{1n}) = \frac{m-n+1}{m}$.

Lemma 11 *Suppose* $\lambda < \frac{\tau-1}{\tau}$ *, then* $\lim_{n\to\infty} h_{1n} > h^*$ *whenever this limit exists.*

Proof. Take λ such that $\lambda m < n$ (or $\lambda < \frac{1}{\tau}$). Then $\lim_{n\to\infty} h_{1n} \geq h^*$ since no regular worker invests less than h^*. Suppose to the contrary then that $\lim_{n\to\infty} h_{1n} = h^*$. Take $h' > h^*$. There is some n large enough that $h_{1n} < h'$ for all $n' > n$. This implies that for all $n' > n$ there is some strictly positive probability that a worker who invests h' will match with a partner whose investment is at least k^*. The expected quality of the partner of a worker who invests h' must be strictly positive. A worker who invests h^* will rank behind all rational workers with probability 1, and so for large enough n, the expected quality of a worker who invests h^* must be arbitrarily close to zero. Since h^* must be part of the support of the symmetric equilibrium strategy by Lemma 6, this presents a contradiction since h' represents a profitable deviation. ■

We now proceed to provide an upper bound on $\lim_{n\to\infty} h_{1n}$ that depends only on certain characteristics of the weak limit of workers' equilibrium strategy as n goes to infinity. This upper bound will impose an upper limit on firms' equilibrium payoff which must ultimately coincide with the lower limit implicitly established in Lemma 11. We then show that this same characteristic of the weak limit of workers' equilibrium strategies imposes a second restriction on firms' equilibrium payoffs that will typically be inconsistent with the first.

First some simple properties of workers' equilibrium strategies in large games.

Lemma 12 *Let* $h^* < h' < \lim_{n\to\infty} h_{1n}$*. Then along any sequence for which* G_n *converges weakly,* $\lim_{n\to\infty} G_n(h') = \lim_{n\to\infty} \frac{m-n+1}{m} = \frac{\tau-1}{\tau}$

Proof. $G_n(h') \leq G_n(h_{1n})$ for n large enough. So suppose to the contrary that $\frac{\tau-1}{\tau} > \lim_{n\to\infty} G_n(h')$. By Lemma 10 $\hat{G}_n(h')$ can be made arbitrarily close to $G_n(h') < \frac{\tau-1}{\tau}$ with arbitrarily high probability by taking n large enough. This means that the probability $\hat{G}_n(h')$ exceeds $\frac{n-n+1}{m} \to \frac{\tau-1}{\tau}$ can be made arbitrarily close to zero. But then any worker who invests h' will find that the $n-m+1^{st}$ order statistic of the other workers investments will strictly exceed his investment with probability close to one, so the probability such a worker will find a partner is close to zero. The expected return to investment is then arbitrarily close to zero, which implies that h' cannot lie in the equilibrium strategy for large enough n. Then by Lemma 6, h_{1n} cannot lie in the support of the equilibrium strategy for large enough n. A contradiction. ■

This lemma shows that the equilibrium strategy for workers tends toward one which has an atom at h^*. The size of this atom is exactly the extent of the excess supply of workers. The limit equilibrium strategy then has a flat segment along which the probability with which any worker invests in this flat segment shrinks to zero.

Lemma 13 *Let h'' be any point such that h'' is in the support of G_n for large enough n and $\lim_{n\to\infty} G_n(h'') > \frac{\tau-1}{\tau}$. Let u_n^e be the* equilibrium *payoff of workers. Then $u(h'') + v^* \leq \lim_{n\to\infty} u_n^e$*

Proof. Again, suppose the contrary. Again using Lemma 10, the probability that $\hat{G}(h'') < \frac{\tau-1}{\tau}$ can be made arbitrarily close to zero - so any worker who invests h'' will match with some firm with probability arbitrarily close to one. Since no firm invests less than k^* the expected investment of the workers partner in equilibrium must then be at least v^*, a contradiction. ■

Since the set of probability distributions on $[h^*, h^{**}]$ is closed under the weak topology, we can assume that there is a subsequence along which $\lim_{n\to\infty} G_n(\cdot)$ has a weak limit which is itself a distribution function. Let G denote this weak limit. Let $h_0 \equiv \inf\left\{h : G(h) > \frac{\tau-1}{\tau}\right\}$. For each $h \in [h^*, h_0]$ define $\kappa(h)$ to be the solution to $u(h) + k = u(h^*)$.

Lemma 14 *Suppose that $\lambda < \frac{\tau-1}{\tau}$. Then $\lim_{n\to\infty} h_{1n} \leq \frac{1}{\kappa(h_0)} \int_{h^*}^{h_0} hd\kappa(h)$.*

The proof of Lemma 14 is subtle. The result itself is not intuitive. It establishes that the distribution function of h_{1n} is stochastically dominated by a distribution function which looks like the workers indifference curve through $u(h^*)$ on the interval $[h^*, h_0]$. The important point about the Lemma is that this bound depends *only* on the shape of workers' indifference curves, and on the point h_0 where workers focus their investments in the limit.

Proof. (of Lemma 14) If since $\lambda < \frac{\tau-1}{\tau}$, $\lambda m < m - n + 1$ for large n. Then the actual number of poor workers will be strictly less than $m - n + 1$ with arbitrarily high probability when n is large enough. So a regular worker who invests h^* will remain unmatched with arbitrarily high probability as n gets large. It follows that $\lim_{n\to\infty} u_n^e = u(h^*)$. Let $\hat{k}_n(h)$ be the (random) realized quality of the firm with whom a worker matches if he invests h in the market with n firms and $m = \tau n$ workers. Observe that $\mathbb{E}\hat{k}_n(h) = \{k : u(h) + k = u_n^e\} \geq \kappa(h)$.

Let $h'' > h_0$. Then $\liminf_{n\to\infty} G_n(h'') > \frac{\tau-1}{\tau}$.[14] Suppose $\liminf_{n\to\infty} G_n(h'') = \frac{\tau-1}{\tau} + \delta$. Then by Lemma 10, $\hat{G}_n(h'') > \frac{\tau-1}{\tau} + \frac{2\delta}{3}$ with arbitrarily high probability as n gets large. By assortative matching, the investment h'' must result in a match with a firm whose investment exceeds the investments of the proportion $\frac{2\delta}{3}$ of the other firms, again with arbitrarily high probability as n gets large.

On the other hand consider a worker who invests $h < h_0$. Now by standard properties of weak convergence, $\limsup G_n(h_0) \leq G\left(h_0 - \frac{h_0-h}{2}\right) \leq \frac{\tau-1}{\tau} + \frac{\delta}{3}$. So $\hat{G}_n(h) \leq \frac{\tau-1}{\tau} + \frac{\delta}{3}$ with arbitrarily high probability. Assortative matching implies that the investment h must result in a match with a firm whose investment is larger than no more than the proportion $\frac{\delta}{3}$ of the other firms, with high probability. Let $\hat{h}_{1n}$ denote the $m - n^{th}$ highest investment among the 'other' $m - 1$ workers. A worker who invests h will match with some firm if $\hat{h}_{1n} < h$ (by Lemma 5 $\hat{h}_{1n} = h$ occurs with zero probability). If $\hat{h}_{1n} > h$ the worker will not be matched with any firm. Since the event $\hat{h}_{1n} < h$ must occur with probability strictly bounded above zero for all n (in order for an investment $h > h^*$ to produce expected payoff u_n^e for all n) it must be the case that *conditional on the event* $\hat{h}_{1n} < h$, $\hat{G}_n(h) \leq \frac{\tau-1}{\tau} + \frac{\delta}{3}$ with arbitrarily high probability. Thus as n increases,

$$\mathbb{E}\hat{k}_n(h) = \Pr\left\{\hat{h}_{1n} \leq h\right\} \mathbb{E}\left\{\hat{k}_n(h) | \hat{h}_{1n} \leq h\right\}$$

[14]This is because there are points $h''' < h''$such that $G(h''') > \frac{\tau-1}{\tau}$. Then since $G_n(h^*) = 0$ by Lemma 5 for all n, $\liminf G_n((h^*, h'')) \geq G((h^*, h'')) \geq G((h^*, h''']) = G([h^*, h'''])$.

$$\leq \Pr\left\{\hat{h}_{1n} \leq h\right\} \mathbb{E}O_{F_n}\left(\left(\frac{\tau-1}{\tau}+\frac{\delta}{3}\right)n\right)$$

$$\leq \Pr\left\{\hat{h}_{1n} \leq h\right\} \mathbb{E}O_{F_n}\left(\left(\frac{\tau-1}{\tau}+\frac{2\delta}{3}\right)n\right)$$

$$\leq \Pr\left\{\hat{h}_{1n} \leq h\right\} \mathbb{E}\left\{\hat{k}_n\left(h''\right)\right\}$$

The function $\mathbb{E}\hat{k}_n(\cdot)$ is a sequence of indifference curves for workers converging pointwise to the indifference curve through h^*. Since this family of indifference curves is assumed to be eqi-continuous on the interval $[h^*, h^{**}]$ this sequence converges uniformly. Thus

$$\mathbb{E}\hat{k}_n(h) \leq \Pr\left\{\hat{h}_{1n} \leq h\right\} \mathbb{E}\left\{\hat{k}_n(h_0)\right\}$$

for large enough n for each $h^* \leq h < h_0$. So pointwise

$$\lim_{n\to\infty} \Pr\left\{\hat{h}_{1n} \leq h\right\} \geq \lim_{n\to\infty} \frac{\mathbb{E}\hat{k}_n(h)}{\mathbb{E}\left\{\hat{k}_n(h_0)\right\}}$$

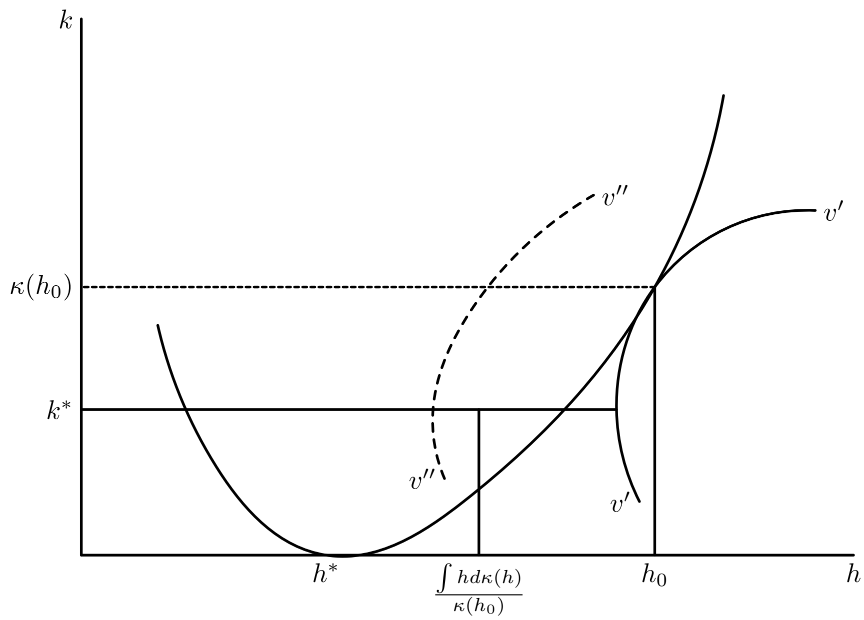
The function $\Pr\left\{\hat{h}_{1n} \leq h\right\}$ describes a probability distribution for the $m-n^{th}$ highest investment among $m-1 = \tau n - 1$ workers. For large n, this is also the probability distribution over returns for the firm who invests k^*. Take any subsequence along which this sequence of distribution functions has a weak limit. This weak limit is first order stochastically dominated by the distribution function

$$\frac{\mathbb{E}\hat{k}(h)}{\mathbb{E}\hat{k}(h_0)}$$

So we have the following upper bound on the limiting value of the return to the firm who invests k^*. It is

$$\int_{h^*}^{h_0} hd\frac{\mathbb{E}\hat{k}(h)}{\mathbb{E}\hat{k}(h_0)} = \int_{h^*}^{h_0} hd\frac{\kappa(h)}{\kappa(h_0)}$$

■



Lemmas 12 and 14 together make it possible to show the basic result in this section. It may help to use the following figure to understand the argument.

Since the discussion focuses on workers strategies in this section, the worker's investment is described along the horizontal axis. The U-shaped curve is the indifference curve along which a worker gets the same payoff as he does when he is unmatched. As described above, each workers equilibrium payoff must tend to this as the number of traders in the market gets large.

In the picture, h_0 is chosen so that it is equal to the efficient investment for workers (tangency between the firms and workers indifference curves). Lemma 12 shows that when there are many traders, workers will put a lot of probability weight near $h^*$and not much weight in a non-degenerate interval above h^*. If the equilibrium is to be efficient, we want workers to assign a lot of probability weight to investments near the efficient investment h_0. In the limit, the strategy should consist of a pair of atoms, one at h^* to eliminate the excess supply and one at h_0 the efficient investment. If that happens firms payoff will tend to one that supports the indifference curve $v'v'$.

The Nash investment k^* is in the support of firms' equilibrium strategy for all n. As the equilibrium payoff must be the same everywhere in the support of the equilibrium strategy, a firm who invests k^* must end up with the same payoff as a firm who invests near $\kappa(h_0)$.Since the expected quality of the firms' partner when it invests k^* is equal to the expected value of the $m-n+1^{st}$ order statistic among firms investments, we can apply Lemma 14 to show that the expectation of this order statistic tends as n gets large to something that cannot exceed $\int h d\kappa(h) / \kappa(h_0)$. Without providing an explicit algebraic example, it is clear from the picture that this bound typically won't support the efficient payoff for firms. The bound is given by averaging h using the continuous distribution function whose value is given for each value of h by the ratio of $\kappa(h)$ to $\kappa(h_0)$ (note that this bound depends on h_0 and the shape of the workers indifference curve, but is wholly independent of the shape of the firms indifference curve away from the tangency. A counter example to efficiency is simply generated by making the firms indifference curve steep).

We conclude that the simultaneous pre-marital investment game doesn't support efficient investment when the number of traders is large. Rather than probing in more detail for the nature of the equilibrium strategies, we move instead to the sequential game.

5 Sequential Investment Game

In this section, we revert to the assumption that all workers are regular with probability 1. Firms choose their investments first, then workers follow suit after observing these. Nothing about the equilibrium of the workers second stage game changes, except that in the calculation of the symmetric equilibrium strategy, the expectation of the order statistics for firms is replaced by the actual investments that the firms undertook in the first stage. It is immediate from Lemma 7 that a unique symmetric equilibrium exists for each array of investments by firms in the first stage. Furthermore, by Lemma convex support, the workers equilibrium strategy is convex and contains their Nash investment.

Using previous notation, the payoff to firms in the first stage of the game is given by

$$V(k_i, k_{-i}) \equiv$$

$$\sum_{j=1}^{n} R_j(k_i, k_{-i}) \{v(k_i) + \mathbb{E}_{G_n} O_{m-n+j}(h)\} \quad (9)$$

where (recall) $R_j(k_i, k_{-i})$ is the probability that the firm who invests k_i is ranked j^{th} lowest among firms by the matching process, when other firms investments are given by k_{-i}. Since firms all have the same type, the actual matching process is independent of the investments of the workers. The expectation of the order statistics for firms investments is taken using the distribution function G_n associated with the symmetric equilibrium for the firms second stage game that follows investments k_i and k_{-i}. By Corollary 8, this equilibrium varies continuously as the firm adjusts k_i. Since each of the order statistics is continuous in the array of investments, the expected quality of the firms partner is actually continuous in his or her own investment. The probability function R_j however, is not continuous, and the firm can still discretely increase his or her rank by raising investment slightly.

Our focus here is on symmetric pure strategy equilibria, so we consider only three cases. One possibility is that all firms offer the same investment, say k_0. The second, all but one firm offer k_0, the other firm (the deviator) invests $k_1 < k_0$. The third possibility is that the deviator raises investment above k_0. Let $v_n^e(k', k_0)$ denote the payoff in the continuation equilibrium with n firms (that is, when workers adopt their equilibrium strategy G_n) to a firm who deviates to k' when all other firms use investment k_0.

First note that in the case considered here $\lambda = 0$, which satisfies the conditions for Lemma 12 and 14. Furthermore, observe that when $\lambda = 0$, then $u_n^e = u(h^*)$ for all n, by Lemma 6. So we conclude that as the number of traders gets large, then in each of the cases discussed above, the equilibrium strategy for workers converges to one that has an atom at h^* of size $\frac{\tau-1}{\tau}$. As before, take any weak limit G of the sequence of equilibrium strategies for workers. Let $h_0 = \inf\left\{h : G(h) > \frac{\tau-1}{\tau}\right\}$.

Lemma 15 *Let* $h'' = \max\{x : x \in \operatorname{supp} G_n\}$. *Then* $h'' = \{h : \max[k_0, k_1] = \kappa(h)\}$.

Proof. A worker who invests h'' will be the worker with the largest investment with probability 1 and will therefore match with the firm who has the highest investment. This will be $\max[k_0, k_1]$. Since this investment must yield the same expected payoff to workers as every other investment in the support of the equilibrium strategy, i.e., $u(h^*)$ this highest investment must equal $\kappa([k_0.k_1])$. ■

Lemma 16 $h_0 = \{h : \kappa(h) = k_0\}$

Proof. Let $h' > h_0$. Then there is some large enough n such that $G_{n'}(h) > \frac{\tau-1}{\tau}$ for all $n' > n$. But then by Lemma 10 the probability is arbitrarily close to one that the investment h' strictly exceeds the $m-n^{th}$ (or the $m-n+1^{st}$) investment of the other workers, so that a worker who invests h' will be almost sure to match with a firm whose investment is at least k_0 when n is large enough. If h_0 doesn't satisfy the condition given above, then the payoff to workers who invest close to h_0 will differ from $u(h^*)$ for n large enough. ■

The implication of the last two lemmas is that the equilibrium strategy for workers must converge weakly to a degenerate strategy with atoms at h^* and h_0.

It is now straightforward to compute the payoff of each firm when all firms offer the same investment k_0.

Lemma 17 $\lim_{n\to\infty} v_n^e(k_0,k_0) = v(k_0) + h_0$ *where* h_0 *is given by Lemma 16*

Proof. Using 9 it is

$$\sum_{j=1}^{n} R_j(k_i,k_{-i})\{v(k_0) + \mathbb{E}_{G_n} O_{m-n+j}(h)\}$$

$$= \sum_{j=1}^{n} \frac{1}{n}\{v(k_0) + \mathbb{E}_{G_n} O_{m-n+j}(h)\}$$

$$= \sum_{j=1}^{n} \frac{1}{n}\left\{v(k_0) + \mathbb{E}_{G_n}\left\{h : \hat{G}_h(h) = \frac{m-n+j}{m}\right\}\right\}$$

$$= v(k_0) + \mathbb{E}_{G_n} \sum_{j=1}^{n} \frac{1}{n}\left\{h : \hat{G}_h(h) = \frac{m-n+j}{m}\right\}$$

$$= v(k_0) + \mathbb{E}_{G_n} \int_{h^*}^{h_0} h \frac{d\hat{G}_n(h)}{\hat{G}(h_0) - \hat{G}(h^*)}$$

Now apply Lemma 10 and that fact almost sure uniform convergence of $\hat{G}_n$ to G_n and weak convergence of G_n to G implies almost sure weak convergence of $\hat{G}_n$ to G to get this equal to

$$v(k_0) + h_0$$

■

Now for any k_0, if a firm deviates downward, it will surely be matched with the worker whose investment ranks $m - n + 1^{st}$. This will be true no matter how low the firms investment, so if a firm wants to deviate downward, it might as well revert to its Nash investment. Then in the limit, the maximal profit to a downward deviation is

$$v(k^*) + \lim_{n\to\infty} h_{1n} \leq v(k^*) + \frac{1}{\kappa(h_0)} \int_{h^*}^{h_0} h d\kappa(h)$$

by Lemma 14. Then as before if $\frac{1}{\kappa(h_0)} \int_{h^*}^{h_0} h d\kappa(h)$ is not too close to h_0 this will be lower than the initial profit level and in the limit, the deviation is unprofitable. Since we can get arbitrarily close to these limits by taking n large enough, the following must be true

Lemma 18 *Let k_0 be any common investment for firms and let*

$$h_0 = \{h : u(h^*) = u(h) + k_0\}$$

Then if $v(k^*) + \frac{1}{\kappa(h_0)} \int_{h^*}^{h_0} h d\kappa(h) < v(k_0) + h_0$, then there is some n large enough such that no firm will find it profitable to unilaterally deviate to any $k_1 < k_0$.

The sequential pre-marital investment game behaves a little bit like the Bertrand pricing problem when there are many workers and firms. Firms who try to undercut (in their investments) suffer a discontinuous loss in profits.

We now consider whether firms can gain by raising investment. One unique feature of the sequential game is that a firm who deviates by raising investment above that offered by all the other firms, she shifts the entire support of the equilibrium strategies of the workers by Lemma 15 above. If there is a deviation to $k_1 > k_0$, let h_1 be the top of the support of workers' equilibrium investment strategies, as given by Lemma 15. For each $h \in (h_0, h_1]$ $\lim_{n\to\infty} G_n(h) = 1$, for otherwise, by Lemma 10, the probability will get arbitrarily close to 1 as n gets large than some other worker will have a higher investment. The payoff associated with any such investment must then converge to $u(h)+k_0 < u(h_0)+k_0$. So there is no guarantee that a firm who deviates to k_1 will match with a worker whose expected investment is h_1. Methods similar to those used above can be applied to limit the return to an upward deviation.

Lemma 19 *Suppose* $k_1 > k_0$. *Then* $\lim_{n\to\infty} v_n^e(k_1, k_0) \leq v(k_1)+\int_{h_0}^{h_1} hd\frac{\hat{k}(h)-k_0}{k_1-k_0}$

Proof. Consider any investment by workers between h_0 and $h_1$which is the new upper bound induced by a deviation to k_1. A worker who invests h in this region gets payoff that cannot exceed

$$Q_n(h)\,k_1 + (1 - Q_n(h))\,k_0$$

where $Q_n(h)$ is the probability that the highest investment of the other $m-1$ workers is less than or equal to h given the equilibrium strategy G_n. (No higher follows because the worker could end up without a partner when he is not highest, though this happens with diminishing probability as n gets large). Reasoning as above, this means that

$$\hat{k}(h) \leq Q_n(h)\,k_1 + (1 - Q_n(h))\,k_0$$

or

$$Q_n(h) \geq \frac{\hat{k}(h) - k_0}{k_1 - k_0}$$

The expected quality of the partner of the firm investing k_1 is then less than or equal to

$$\int_{h_0}^{h_1} \left\{ G_n(h)\,h + \int_{h}^{h_1} h' dG_n(h') \right\} d\frac{\hat{k}(h) - k_0}{k_1 - k_0}$$

Since $G_n(h)$ is converging almost everywhere to 1 on the interval $[h^*, h_1]$ by the argument above, this integral has limit

$$\int_{h_0}^{h_1} hd\frac{\hat{k}(h) - k_0}{k_1 - k_0}$$

Notice that this is strictly less than h_1. ■

Once again, this lemma shows that the return to a firm who deviates to $k_1 > k_0$ is *strictly* bounded below h_1 by an amount that depends on the shape of workers' indifference curves. This leads to the main theorem in this section.

Theorem 20 *Let k_0 be such that*

$$v(k^*) + \frac{1}{\kappa(h_0)} \int_{h^*}^{h_0} h d\kappa(h) < v(k_0) + h_0$$

where $h_0 = \{h : u(h) + k_0 = u(h^*)\}$ and for all $k_1 > k_0$,

$$v(k_1) + h_1 < v(k_0) + h_0$$

where $h_1 = \{h : u(h_1) + k_1 = u(h^*)\}$. Then

$$\lim_{n\to\infty} v_n^e(k_1, k_0) < \lim_{n\to\infty} v_n^e(k_0, k_0)$$

for all k_1.

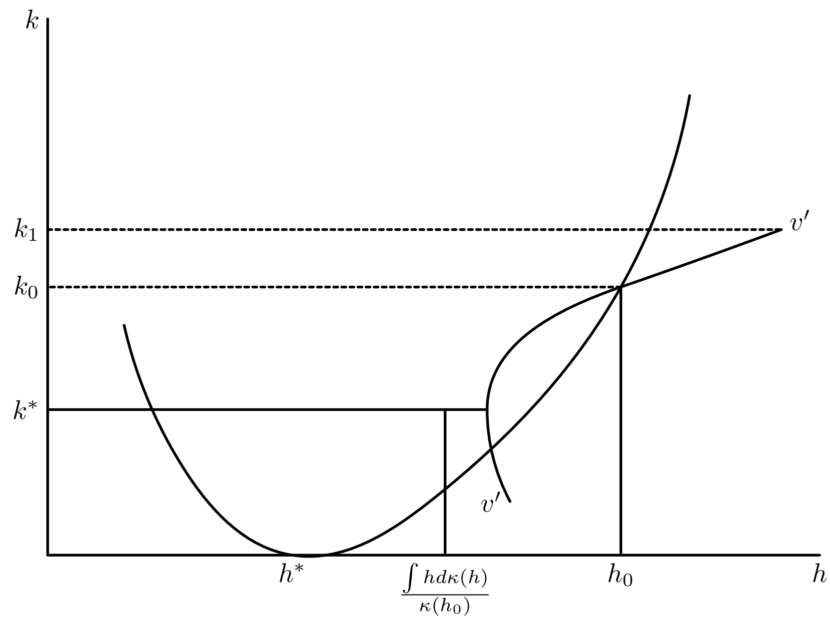
Proof. Follows immediately from the last two lemmas. ■

We can now show the implication of this theorem with the help of Figure 5

The figure represents a situation where all firms choose investment k_0. Worker use mixed investment strategies which must put give them the utility $u(h^*)$. The curve along which workers get this payoff is the curve that is convex upward and touches the horizontal axis at h^* in the Figure. By Lemma 17, each firms payoff will tend to $v(k_0) + h_0$. The indifference curve for firms through this point is given by the curve $v'v'$ in the Figure.

Any firm who deviates downward gets a payoff that is no higher than $v(k^*) + \frac{1}{\kappa(h_0)} \int_{h^*}^{h_0} h d\kappa(h)$ which is labelled in the figure. The figure is drawn according to the restrictions given by Theorem 20. As can be seen, this puts the deviating firm on a lower indifference curve and is unprofitable. By Lemma 19, a firm who unilaterally deviates up to k_1 should expect a partner whose expected investment is something less than h_1 in the limit as n gets large. Since the indifference curve for firms lies below the indifference curve for workers to the right of h_1 this must eventually reduce the deviating firms profits below what they would have been if she had simply offered k_0.

It is correct that for the deviation k_1 shown in the Figure, and from Lemma 18 that there will be some finite n for which it won't pay a deviator



to deviate downward by any amount, or upward to k_1. This does not show that k_0 is an exact equilibrium for large n for a couple of reasons. The obvious one is that the size of n needed to get the strict inequality depends on the deviation k_1 chosen. The other is that the indifference curve that firms attain in equilibrium for very large n lies slightly to the left of the curve drawn in the picture. These results do not rule out the possibility that for every n there is some *small* upward deviation that increases the firms profits. Since the deviation is small and everything is continuous, this is enough to verify that k_0 is an approximate equilibrium for large enough n.

There will also be a variety of outcomes like the one depicted for which it can be shown that the common investment by firms is an approximate equilibrium. All that needs to hold at k_0 is that the indifference curve for the firm lies below the indifference for workers to the right of k_0. This will be true, if for example, we choose k_0 to be the efficient investment level. In this sense, the sequential game supports the hedonic outcome. Yet it supports outcomes where investment is inefficiently large as well.

6 Appendix - Proofs of Theorems

Proof. (Proof of Theorem 1). To verify the existence of a mixed strategy equilibrium, we take the approach of Reny (Reny 1999) and show that the mixed extension of the investment game is *reciprocally upper semi-continuous* and payoff secure. Observe first that the strategy spaces are assumed to be compact connected intervals in $\mathbb{R}^+$. So the investment game is a compact game with metric strategy spaces.

Let $\triangle_i$ be the compact set of regular countably additive probability measures on (the Borel sets of) the set $i = K, H$. The mixed extension of the first stage game is the game in which the firms' and workers' strategy spaces are given respectively by $\triangle_F$ and $\triangle_W$. for any vector of probability measures $\mu = \{\mu_1, \ldots \mu_n, \mu_{n+1}, \ldots \mu_{n+m}\}$ the payoffs are given by

$$\tilde{V}_i(\mu) \equiv \int V_i(x_i, k, \mu_{-i})\, d\mu_i$$

for firms, with a similar expression $\tilde{U}_i(\mu)$ for workers.

Given vectors $k = (k_1, \ldots k_n)$ and $h = (h_1, \ldots h_m)$, the sum of the payoff functions is

$$S(k_1, \ldots h_n) = \sum_{i=1}^{n} v\left(x_i, k_i, h_{\pi(i)}\right) + u\left(y_{\pi(i)}, h_{\pi(i)}, k_i\right)$$

where the matching function depends on the entire vector of investments. If $k_i \neq k_j \forall j \neq i$, then the sum of the payoffs is continuous in k_i because a small change in i's investment will not change his partner, and both the payoff function of firm i and the payoff function of worker $\pi(i)$ are continuous in i's investment. So suppose that $k_i = k_j$ for some j.

If upper semi-continuity fails at k_i, then there is an $\varepsilon > 0$ and a sequence $\{k^\tau\} \to k_i$ such that $\lim_{k^\tau \to k_i} S(k^\tau, k_{-i}, h) > S(k_i, k_{-i}, h) + \varepsilon$. Since the number of firms is finite, we can choose the sequence $\{k^n\}$ such that trader i's rank in the array of investments of firms is constant along the sequence. By the matching regularity condition, this implies that the matching function is constant for each element of the sequence. Let π' denote this constant matching function. Then by the continuity of the value functions

$$\sum_{i=1}^{n} v\left(x_i, k_i, h_{\pi'(i)}\right) + u\left(y_{\pi'(i)}, h_{\pi'(i)}, k_i\right) >$$

$$\sum_{i=1}^{n} v\left(x_i, k_i, h_{\pi(i)}\right) + u\left(y_{\pi(i)}, h_{\pi(i)}, k_i\right)$$

Since π' satisfies the assortative matching condition for investments (k_i, k_{-i}, h), π violates the second matching regularity condition. We conclude that the sum of the payoffs is upper semicontinuous. Then by (Reny 1999), the mixed extension of the investment game is reciprocally upper semicontinuous.

The mixed extension of any game is *payoff secure* if for every array of strategies μ, and each $\varepsilon > 0$, each player i has a strategy $\overline{\mu}_i$ such that

$$\tilde{U}_i\left(\overline{\mu}_i, \mu'_{-i}\right) \geq \tilde{U}_i\left(\mu\right) - \varepsilon$$

for all μ'_{-i} in some weak neighborhood of μ_{-i}. We show that the mixed extension of the investment game is payoff secure.

Fix strategies μ^*_{-i} and let $v^* = \sup_k V_i\left(x_i, k, \mu^*_{-i}\right)$. Let k^* be some investment level for i such that

$$\left|v^* - V_i\left(x_i, k^*, \mu^*_{-i}\right)\right| < \frac{\varepsilon}{2}$$

Observe that the underlying utility function $v\left(x_i, \cdot, \cdot\right)$ is continuous, thus uniformly continuous on compact sets. Then there is a δ such that $v\left(x_i, k^* + \delta, h'\right) > v\left(x_i, k^*, h\right) - \frac{\varepsilon}{2}$ for all $h \in H$. Let $\overline{R}\left(k, k_{-i}\right) = 1 + \#\left\{j : k_j < k\right\}$. Since $\overline{R}\left(k, k_{-i}\right)$ is the worst ranking that i can have when he invests k, then $V_i\left(x_i, k^*, \mu^*_{-i}\right)$ satisfies

$$\int \cdots \int \sum_{j=1}^{m} R_j\left(k^* + \delta, k_{-i}, h\right) v\left(x_i, k^* + \delta, h_{(j+n-m)}\right) d\mu_{-i} \geq$$

$$\int \cdots \int v\left(x_i, k^* + \delta, h_{\left(\overline{R}(k^*+\delta, k_{-i})\right)}\right) d\mu_{=i} >$$

$$\int \cdots \int v\left(x_i, k^*, h_{\left(\overline{R}(k^*+\delta, k_{-i})\right)}\right) d\mu_{-i} - \frac{\varepsilon}{2} \geq$$

$$\int \cdots \int \sum_{j=1}^{m} R_j\left(k^*, k_{-i}, h\right) v\left(x_i, k^*, h_{(j+n-m)}\right) d\mu_{-i} - \frac{\varepsilon}{2} \qquad (10)$$

for *any* array of strategies μ_{-i} used by the other traders.

Now consider the expression

$$\int \cdots \int v\left(x_i, k^*, h_{\left(\overline{R}(k^*+\delta, k_{-i})\right)}\right) d\mu^*_{-i} \tag{11}$$

Let $\mathcal{P}(H)$ and $\mathcal{P}(K)$ be the sets of probability measures on H and K respectively endowed with the weak topology. We wish to show that there is a neighborhood (in the product topology) of μ^*_{-i} such that

$$\int \cdots \int v\left(x_i, k^*+\delta, h_{\left(\overline{R}(k^*+\delta, k_{-i})\right)}\right) d\mu_{-i} >$$

$$\int \cdots \int v\left(x_i, k^*+\delta, h_{\left(\overline{R}(k^*+\delta, k_{-i})\right)}\right) d\mu^*_{-i} - \frac{\varepsilon}{2} \tag{12}$$

for all μ_{-i} in this neighborhood. We show that for any sequence $\left\{\mu^t_{-i}\right\}$ that converges weakly to μ^*_{-i} in the product topology

$$\liminf \int \cdots \int v\left(x_i, k^*+\delta, h_{\left(\overline{R}(k^*+\delta, k_{-i})\right)}\right) d\mu^t_{-i} \geq$$

$$\int \cdots \int v\left(x_i, k^*+\delta, h_{\left(\overline{R}(k^*+\delta, k_{-i})\right)}\right) d\mu^*_{-i} - \frac{\varepsilon}{2} \tag{13}$$

The order statistics $h_{(\cdot)}$ are all continuous functions, and v is continuous. Since H is compact, v is a uniformly continuous function of each worker's investment. Then by the standard property of weak convergence

$$\lim \int \cdots \int v\left(x_i, k^*+\delta, h_{\left(\overline{R}(k^*+\delta, k_{-i})\right)}\right) d\mu^*_{-ij} d\mu^t_j =$$

$$\int \cdots \int v\left(x_i, k^*+\delta, h_{\left(\overline{R}(k^*+\delta, k_{-i})\right)}\right) d\mu^*_{-i}$$

and (13) follows.

For any *firm* i take any other firm j and define $\overline{R}_{-j}(k_i, k_{-i}) = 1 + \#\left\{i' \neq j : k_i > k_{i'}\right\}$. Let μ_j be an arbitrary mixed strategy for firm j. Write (11) as

$$\dot{\mu}_j(k_i) \int \cdots \int v\left(x_i, k^*+\delta, h_{\left(\overline{R}(k^*, k_{-i})+1\right)}\right) d\mu^*_{-ij}(k) +$$

$$\left[1-\dot{\mu}_j\left(k_i\right)\right]\int\cdots\int v\left(x_i,k^*+\delta,h_{\left(\overline{R}\left(k^*,k_{-i}\right)\right)}\right)d\mu^*_{-ij}\left(k\right) \tag{14}$$

where $\dot{\mu}^*_j\left(k_i\right)$ is the probability that firm j chooses an investment strictly less than k_i. The function $\dot{\mu}_j$ describes the probability weight assigned to an open set. By standard properties of weak convergence (Billingsley 1999), $\liminf \dot{\mu}^t_j\left(X\right)\geq\dot{\mu}^*\left(X\right)$ for each open X, and so

$$\liminf \dot{\mu}^t_j\left(k_i\right)\int\cdots\int v\left(x_i,k^*+\delta,h_{\left(\overline{R}\left(k^*,k_{-i}\right)+1\right)}\right)d\mu^*_{-ij}\left(k\right)+$$

$$\left[1-\dot{\mu}^t_j\left(k_i\right)\right]\int\cdots\int v\left(x_i,k^*+\delta,h_{\left(\overline{R}\left(k^*,k_{-i}\right)\right)}\right)d\mu^*_{-ij}\left(k\right)$$

because of the fact that $h_{\left(\overline{R}\left(k^*,k_{-i}\right)+1\right)}\geq h_{\left(\overline{R}\left(k^*,k_{-i}\right)\right)}$. Combining these last two results gives (13). Combining (10) and (13) gives that for any ε,there is a δ and a weak neighborhood of μ^*_{-i} such that

$$\int\cdots\int\sum_{j=1}^{m}R_j\left(k^*+\delta,k_{-i},h\right)v\left(x_i,k^*+\delta,h_{(j+n-m)}\right)d\mu_{-i}\geq$$

$$\int\cdots\int\sum_{j=1}^{m}R_j\left(k^*,k_{-i},h\right)v\left(x_i,k^*,h_{(j+n-m)}\right)d\mu^*_{-i}-\varepsilon$$

which is payoff security in the mixed extension. ■

Proof. (of Lemma 2) For part 1, observe that any firm whose current investment is strictly less than their Nash investment will want to increase investment even if this does not improve the quality of their partner. So this follows from assortative matching. To show part 2, we use a geometric argument. Begin with a point k^* that lies in the support of firm i's equilibrium strategy. Consider the iso-payoff line for firm i through the point $\left(k^*,\overline{r}\left(k^*\right)\right)$. Let $IP^*_j=\left\{\left(k',h'\right):\tilde{v}\left(x_i,k'\right)+h'=\tilde{v}\left(x_i,k^*\right)+\overline{r}\left(k^*\right)\right\}$ denote this iso-payoff line, and note that this line coincides the payoff that firm i attains in equilibrium.

Firm i will never choose an investment that lies to the right of the intersection of this iso-payoff line and the function $\overline{r}_{ij}\left(k\right)$.The reason is that the expected quality $\overline{r}_{ij}\left(k\right)$ assumes that i invests more than j for sure. So moving further to the right along this line cannot put i on a higher iso-payoff line no matter what strategy j is playing.

Let $(k^+, r^+) = \sup\{(k', h') : (k', h') \in IP^*; (k', h') = (k', \overline{r}(k'))\}$. Firm j can do no worse in equilibrium (but may do better) than to invest k^+ and receive a return r^+ since in equilibrium he must surely be investing more than firm i if he does this. By the single crossing condition, the iso-payoff line IP_j^* that firm j attains in equilibrium must then lie everywhere above IP_i^* to the left of the point (k^+, r^+). In particular, this implies that *if* there is a point k^0 in the support of j's equilibrium strategy that lies somewhere to the left of k^+, then $\overline{r}_{ij}(k) > \underline{r}_{ij}(k)$ and $\mu_i(k) > \mu_j(k)$ for all k in some small enough open neighborhood of k^0 (it must be more likely that j beats i by playing k^0 than the converse in order to support he higher average quality that j obtains. Furthermore since $\overline{r}(k)$ is non-decreasing, this must continue to be true in some open neighborhood of k^0).

Now select any investment, say k, in the convex hull of the support of i's equilibrium strategy. If this investment is in the support of both firms equilibrium strategy, the theorem is proved by the argument above. If the point is not in the support of firm j's equilibrium strategy, then there are two possibilities, either there is a point $k' < k$ such that k' is in the support of firm j's equilibrium strategy or not. If not, then the theorem follows trivially. If so, let k' be the nearest such point. Since j does not use any investment in the interval $[k', k]$ the probability that firm i invests more than firm j is constant along this interval and so $\mu_i(k) \geq \mu_i(k') \geq \mu_j(k') = \mu_j(k)$, and the result is proved. ■

Proof. (of Theorem 7) 8 is an ordinary differential equation. By standard theorems ((Kreider, Kuller, and Ostberg 1968)) for differential equations, it will have a unique solution if it is Lipshitzian, in other words if the derivative of the right hand side of the equation with respect to G is bounded above. Write 8 as

$$G' = \frac{-u'(h)}{\Pi'(G) \cdot \overline{k}}$$

where $\overline{k}$ is the vector of expected order statistics of sellers investments (augmented with zeros in the first $m - n$ positions) and $\Pi'(G)$ is the vector of derivatives of the binomial probabilities $\binom{m-1}{t} G^t (1-G)^{m-1-t}$. Observe that $\Pi(G) \cdot e \equiv 1$ for all G (e is an m vector consisting entirely of 1's) so $\Pi'(G) \cdot e = 0$, and that $\Pi_t'(G) < 0$ for each $t < G(m-1)$; $\Pi_t'(G) = 0$

if $t = G(m-1)$ and $\Pi'(G) > 0$ otherwise. These are standard properties of the binomial distribution function. Also note that the second derivative $\Pi''(G)$ will involve a finite sum of finite terms, so will be finite. Thus (8) will be Lipshitzian if $\Pi'(G)\,\overline{k} > 0$ (for all G in $[0,1]$).

Now

$$\Pi'(G)\cdot\overline{k} \geq$$

$$\sum_{t=0}^{(m-n-1)} \binom{m-1}{t} G^{t-1}(h)\,(1-G(h))^{m-2-t}\cdot 0\cdot (t-G(h)(m-1)) +$$

$$\sum_{t=m-n}^{G(h)(m-1)} \binom{m-1}{t} G^{t-1}(h)\,(1-G(h))^{m-2-t}\,\overline{k}_{(G(h)(m-1)-(m-n-1))}\,(t-G(h)(m-1)) +$$

$$\sum_{t=G(h)(m-1)+1}^{m-1} \binom{m-1}{t} G^{t-1}(h)\,(1-G(h))^{m-2-t}\,\overline{k}_{(G(h)(m-1)-(m-n-1)+1)}\,(t-G(h)(m-1))$$

Write

$$\eta_1 = \sum_{t=0}^{(m-n-1)} \binom{m-1}{t} G^{t-1}(h)\,(1-G(h))^{m-2-t}\,(t-G(h)(m-1)),$$

$$\eta_2 = \sum_{t=m-n}^{G(h)(m-1)} \binom{m-1}{t} G^{t-1}(h)\,(1-G(h))^{m-2-t}\,(t-G(h)(m-1))$$

and observe that

$$\eta_1+\eta_2 = \sum_{t=G(h)(m-1)+1}^{m-1} \binom{m-1}{t} G^{t-1}(h)\,(1-G(h))^{m-2-t}\,(t-G(h)(m-1))$$

This gives

$$\Pi'(G)\cdot\overline{k} \geq$$

$$-\eta_1 \cdot 0 - \overline{k}_{(G(h)(m-1)-(m-n-1))}\eta_2 + \overline{k}_{(G(h)(m-1)-(m-n-1)+1)}\left(\eta_1 + \eta_2\right)$$

$$\geq \eta_1 \left(\overline{k}_{(G(h)(m-1)-(m-n-1)+1)}\right) + \eta_2 \left(\overline{k}_{(G(h)(m-1)-(m-n-1)+1)} - \overline{k}_{(G(h)(m-1)-(m-n-1))}\right)$$

$$\geq \eta_1 k^* + \eta_2 \left(\overline{k}_{(G(h)(m-1)-(m-n-1)+1)} - \overline{k}_{(G(h)(m-1)-(m-n-1))}\right)$$

Since $k^* > 0$, $\Pi'(G) \cdot \overline{k} > 0$ provided η_1 and η_2 are both strictly positive, which is true whenever $G > 0$. To complete the argument, then we need to check the size of this derivative when $G = 0$.

Rewrite the denominator of 8 as

$$\frac{1}{G(h)(1-G(h))}\sum_{t=0}^{m-1}\left(\begin{array}{c} m-1 \\ t \end{array}\right) G^t(h)(1-G(h))^{m-1-t}\,\overline{k}_{t-(m-n-1)}\left(t - G(h)(m-1)\right)$$

Both numerator and denominator go to zero with $G(h)$. By L'Hopitals rule, the limit is given by taking the ratio of the limit of the derivatives of these expression, which are both strictly positive since $\overline{k}_{t-(m-n-1)}$ is not a constant sequence. Hence this derivative is strictly positive.

This guarantees the existence and uniqueness of the solution to 8 with initial value $(h^*, 0)$ on a non-degenerate interval $[h^*, h_1]$ say (the size of this interval depends on the Lipshitz constant that emerges above - if the constant is large the interval is small). Since G has to be positive, there are two possibilities, either $G(h) = 1$ for some $h \leq h_1$ or not. If the latter, apply the same theorem again using initial value $(h_1, G(h_1))$ to define the solution uniquely on an interval $[h^*, h_2]$ and continue in this way until $G(h) = 1$. This must occur for some value of $h < h^{**}$. The reason is that any solution to 8 must satisfy 7. If $G(h^{**}) < 1$ then $u(h^{**}) + K^*(h^{**}) < u(h^{**}) + k^{**}$, a contradiction. This argument extends the solution to the point where $G(h) = 1$ and completes the argument. ■

Almost the same argument applies for firms, though all firms will match in any equilibrium of this kind so the term involving η_1 in the proof that $\Pi'(G)\,\overline{h}$ will not exist. The poor worker assumption ensures that the term $\overline{h}_{(F(k)(n-1)+1)} - \overline{h}_{(F(k)(n-1))}$ which multiplies η_2 will be strictly positive.

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