

Menu Theorems for Bilateral Contracting

Seungjin Han*

McMaster University

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Abstract

This paper studies the bilateral contracting environment where multiple principals negotiate contracts with multiple agents independently. It is shown that equilibrium allocations associated with (pure strategy) perfect Bayesian equilibria relative to any ad hoc set of negotiation schemes can be supported by pure strategy perfect Bayesian equilibria relative to the set of menus. It is also shown that equilibrium allocations associated with all perfect Bayesian equilibria relative to any ad hoc set of negotiation schemes can be supported by correlated equilibria relative to the set of menus, where the set of states is simply the set of feasible probability distributions over payoff-relevant variables. Moreover, equilibrium allocations associated with all equilibria relative to the set of menus persist even if principals use more complex negotiation schemes.

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*Department of Economics, Kenneth Talyor Hall, Room 426, McMaster University 1280 Main Street West, Hamilton, Ontario, Canada, L8S 4M4. Email:hansj@mcmaster.ca. This is a revised version of the first chapter in my PhD dissertation at the University of Toronto. I am deeply indebted to Michael Peters for his insightful guidance and advice. I wish to thank Ettore Damiano, Jeffrey Ely, Martin Osborne, Alessandro Pavan, Carolyn Pitchik, Ilya Segal and seminar participants at the University of Toronto and the 2002 Canadian Economic Theory Conference for numerous helpful comments and discussions. Financial support from Social Sciences and Humanities Research Council of Canada is thankfully acknowledged. All errors are mine alone.

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1 Introduction

Efficiency in markets relies fundamentally on competition. Competing sellers drive down prices and raise quantities, competing auctioneers drive down reserve prices, making it more likely that efficient trades will be consummated. It is natural to expect that these beneficial effects of competition should be independent of the way that sellers compete. So, if many principals compete in incentive contracts, for example, their competition should still ensure that outcomes are close to being efficient.

There are many examples in the literature suggesting that this need not be the case. Klemperer (2002), for example, describes implicit collusion that occurred in British electricity auctions with many competitors that results from the fact that prices were determined in a uniform price auction. Klemperer and Meyer (1989) describe the large number of inefficient equilibria that can occur when sellers compete in supply functions. Even though there are equilibria where multiple lobbyists compete away all their rents (Dixit, Grossman and Helpman 1997), there are also many inefficient equilibria in the simplest common agency problem. The bad equilibria in these problems arise in simple static competitive environments so the implicit collusion that occurs has nothing to do with repeated game effects. It does seem to be related to the nature of the competition between sellers. For example, many of the inefficient equilibria described by Klemperer and Meyer (1989) disappear if sellers are simple Cournot competitors.

The complication that arises in environments where sellers compete in more complex ways has been described in the terminology of mechanism design by McAfee (1993). When principals communicate with agents in a competitive environment, the agents have information about what is happening in the market that principals do not have when they design and offer their incentive contracts. If principals ask their agents about this market information, in other words principals use mechanisms, they may be able to learn when

one of the other principals has deviated from some implicit agreement. This lets them punish deviations and enforce collusive outcomes. In other words, if principals use incentive contracts that are sophisticated enough, they can neutralize the effects of competition.

To overcome the difficulties associated with this, it is important to understand exactly how principals support these collusive outcomes. The recent literature on common agency has provided an important insight into how this is done. Principals offer the single agent menus of alternative incentive contracts. If the agent's preference ordering over the alternatives in these menus depends on what other principals have done, then that is often enough for principals to create the punishments that support collusive outcomes. Theorems in, for example, Martimort and Stole (2002) or Peters (2001) show that all collusive outcomes that principals can support can be understood in exactly this way. Furthermore, from Peters (2001), principals who offer menus of incentive contracts cannot do any better by using more complex mechanisms.

The idea that principals offer an agent menus of alternative incentive contracts from which the agent chooses cannot be extended in any obvious way beyond the common agency framework. When there are multiple agents, mechanisms will typically choose incentive contracts based on the messages of many different agents. There is no way to assign the choice to a single agent, as is the case with simple menus.

The only approach available in the case of multiple agents is the one proposed by Epstein and Peters (1999). They describe a language that agents can use to describe their market information to principals. With this language in place, competition between principals can then be modelled as competition in direct mechanisms. The language that is required to describe type is complex because of the infinite regress involved in characterizing mechanisms.¹ So it is of interest to try to find environments where simpler mechanisms might suffice to understand collusion between principals.

This paper studies the *bilateral contracting environment* where principals can assign an incentive contract for each agent conditional on the message that the agent sends but not

¹In general type spaces are just as complex in the single principal case. The type space is made simple by the common prior assumption. No comparable restriction is yet known for competing principal problem.

on the messages sent by the other agents. Therefore, multiple principals negotiate incentive contracts with multiple agents independently. In this case, it makes sense to think about the principal offering each agent a menu of incentive contracts because an incentive contract assigned to an agent does not depend on the communication between principals and the other agents so the agent can directly choose an incentive contract from a menu.

Prat and Rustichini (2002) first model a multiple agency problem which can be viewed as a simple bilateral contracting game. In their model, multiple principals offer single incentive contracts to multiple agents. An incentive contract in their model specifies monetary payments to an agent conditional on efforts taken by the agent. In the Prat and Rustichini's model, the incentive contract offered to one agent does not depend on whether or not the other agents take the incentive contracts offered to them. So, the game is bilateral contracting. Examples of the Prat and Rustichini's model include lobbying, vertical contracts, and competing first-price auctions among many others.

It is however important that the Prat and Rustichini's model restricts principals to offer only a single incentive contract to each agent. No significant market information is revealed by agents because there is no communication between agents and principals. Furthermore, Prat and Rustichini do not address the question of whether equilibrium allocations in their simple model persist even if principals are allowed to use complex mechanisms.

The bilateral contracting game considered in this paper allows principals to offer agents any complex mechanisms that they like. If a principal is able to offer mechanisms sophisticated enough to induce agents to reveal their market information, the principal can punish other principals' deviations from implicit collusion by changing his contracts. Even in a static game, equilibria relative to complex mechanisms can support many collusive outcomes that are not supported by any equilibria in the Prat and Rustichini's model.

This paper shows that the competition relative to the set of menus in common agency in Martimort and Stole (2002) and Peters (2001) can provide a way to characterize equilibrium allocations relative to any set of mechanisms in the bilateral contracting environment in the presence of externalities among players. First of all, it is shown that equilibrium allocations associated with *pure strategy perfect Bayesian equilibria* relative to any set of mechanisms

can be supported by pure strategy perfect Bayesian equilibria relative to the set of menus. Since pure strategy equilibria are of interest in many applications, this result is especially useful.

The menu theorems for common agency show that perfect Bayesian equilibria relative to the set of menus support equilibria allocations associated with all perfect Bayesian equilibria relative to any set of mechanisms. The theorems are not directly extended into the bilateral contracting environment. With externalities between agents, the optimal incentive contracts in menus and the optimal effort for one agent depend on what this agent believes some other agents do. It is therefore natural to expect multiple continuation equilibria to be associated with menus that principals offer. Principals can correlate different continuation equilibrium allocations by changing the name of their menus. If principals are restricted to use menus, only one continuation equilibrium is assigned at menus that principals offer. Therefore such a coordination is not possible if principals are restricted to use menus. This type of problem does not arise in common agency because a single agent is solely able to decide a continuation equilibrium given menus by choosing incentive contracts in menus offered to her. To illustrate this problem, this paper provides an example of an equilibrium relative to a specific set of mechanisms that cannot be supported by equilibria relative to the set of menus.

This paper proposes *correlated equilibria* relative to the set of menus to support equilibrium allocations associated with mixed strategy equilibria relative to any set of mechanisms. In correlated equilibrium, a state is realized from a probability distribution conditional on a collection of menus offered by principals. A state delivers agents a belief about which equilibrium will occur. Given a state, agents choose their incentive contracts in menus. It is important that an equilibrium allocation is fully characterized by a probability distribution over payoff-relevant variables. If the set of states is the set of all feasible probability distributions over payoff-relevant variables, then any equilibrium allocation relative to any set of mechanisms can be associated with a unique state. It is shown that with this set of states, correlated equilibria relative to the set of menus support equilibrium allocations associated with all perfect Bayesian equilibria relative to any set of mechanisms.

One of the concerns about specifying a set of simple mechanisms in a model of competition is that competition relative to a set of simple mechanisms might generate equilibria that disappear once principals are allowed to use more complex mechanisms. Predictions based on such equilibria are questionable because at least conceptually principals should be able to use any mechanisms they like. This paper shows that equilibrium allocations relative to the set of menus are robust in the sense that the equilibrium allocations persist even if the set of mechanisms is enlarged. Predictions based on any equilibrium relative to the set of menus are robust.

2 Preliminaries

Throughout this paper, a set is assumed to be a compact set unless specified. When a measurable structure is necessary, the corresponding Borel σ - algebra is used. For a set X , $\Delta(X)$ denotes the set of probability distributions on X . For any $x \in \Delta(X)$, $\text{supp } x$ denotes the support of the probability distribution x . For any mapping L from S into Q , $L(S)$ denotes the image of L .

The set of principals is $\mathcal{J} = \{1, \dots, J\}$. The set of agents is $\mathcal{I} = \{1, \dots, I\}$. Each principal j takes an action y^j from a set $Y^j = \times_{s=1}^I Y_s^j$. A typical action taken by principal j is given $y^j = (y_1^j, \dots, y_I^j) \in Y^j$. In many competition models in nonlinear prices, y_i^j is a monetary payment between principal j and agent i . $Y = \times_{t=1}^J Y^t$ is the set of actions that all principals can take.

Each agent i takes an effort e_i from a set E_i . $E = \times_{s=1}^I E_s$ is the set of efforts that all agents can take. The principals observe (possibly only partially) the efforts of all agents. Each principal j writes an incentive contract a_i^j for each agent i from a set A_i^j which specifies y_i^j for each agent i that the principal will take conditional on whatever levels of efforts $e = (e_1, \dots, e_I) \in E$ that the principal can verify to the agent. Let A_i^j be the set of feasible mappings from E into Y_i^j . $\mathcal{A}_i^j = \Delta(A_i^j)$ is the set of random incentive contracts (henceforth just incentive contracts) that principal j can offer to agent i . $\mathcal{A}^j = \times_{s=1}^I \mathcal{A}_s^j$ is the set of incentive contracts that principal j can offer to all agents. $\mathcal{A}_i = \times_{t=1}^J \mathcal{A}_i^t$ is the set

of incentive contracts that all principals can offer to agent i . A typical element $(\alpha_1^1, \dots, \alpha_I^J)$ in $\mathcal{A} = \times_{s=1}^I \mathcal{A}_s$ is called a *collection of incentive contracts* that all principals can offer to all agents.

Agent i has private information about her preferences. This information is parameterized by an element, called a valuation, in a measurable space Ω_i . All agents and all principals share a common prior belief that the elements of $\Omega = \times_{s=1}^I \Omega_s$ are jointly distributed according to some probability distribution F on Ω . Each principal's payoff is given by $v^j : Y \times E \times \Omega \rightarrow \mathcal{R}$ and each agent's payoff is given by $u : Y \times E \times \Omega_i \rightarrow \mathcal{R}$.

2.1 Bilateral contracting

A mechanism for principal j is a collection of message spaces, $\mathcal{C}^j = \times_{s=1}^I \mathcal{C}_s^j$, and a mapping from \mathcal{C}^j into \mathcal{A}^j . \mathcal{C}_s^j for each $s \in \mathcal{I}$ is the set of messages that agent s can send to principal j . Without loss of generality, $\mathcal{C}_i^j = \mathcal{C}$ for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$.

The bilateral contracting environment restricts feasible mechanisms that principals can offer to agents. Contracting is *bilateral* if each principal j offers each agent i a bilateral mechanism (simply mechanism hereinafter)

$$\gamma_i^j : \mathcal{C} \rightarrow \mathcal{A}_i^j$$

that describes incentive contracts for agent i conditional on the messages that agent i sends. Let Γ_i^j be a set of feasible mechanisms that principal j can offer to agent i . $\Gamma^j = \times_{s=1}^I \Gamma_s^j$ is the set of mechanisms that principal j can offer to all agents. A typical element $(\gamma_1^1, \dots, \gamma_I^J)$ in $\Gamma = \times_{t=1}^J \Gamma^t$ is called a *collection of mechanisms* that all principals offer to all agents.

A bilateral contracting game begins when each principal simultaneously offer mechanisms, one for each agent. After seeing the collection of mechanisms offered by principals, each agent simultaneously sends a message to each principal and chooses her effort.² Subsequently, all payoffs are realized.

²As the literature on mechanism design assumes, each principal fully commits himself to inform agents of his negotiation schemes. So, each agent knows mechanisms offered to the other agents when she sends messages.

While bilateral contracting between multiple principals and multiple agents is very often observed in practice, there is very little literature on it. Recently, Prat and Rustichini (2002) consider a *delegation game* under complete information which can fit into bilateral contracting between multiple principals and multiple agents. In the Prat and Rustichini's model, each principal j offer an incentive contract $a_i^j : E_i \rightarrow Y_i^j$ to each agent i , where a typical element y_i^j in Y_i^j is a monetary payment from principal j to agent i . Given contracts, each agent i chooses an effort $e_i \in E_i$ that determines payoffs. An incentive contract a_i^j can be interpreted as a degenerate mechanism $\gamma_i^j : \mathcal{C} \rightarrow \mathcal{A}_i^j$ that satisfies $\gamma_i^j(c_i^j) = a_i^j$ for all $c_i^j \in \mathcal{C}$. The game in which each principal offers a single incentive contract to each agent is therefore a bilateral contracting game.

Examples in the Prat and Rustichini's delegation game include lobbying games, vertical contracts, and competing first-price auctions among many others. In lobbying game, an effort $e_i \in E_i$ chosen by policy maker i (an agent) is a policy. The i th component y_i^j in an action y^j is a monetary contribution made by lobbyist j (a principal). Each lobbyist j offers each policy maker i a monetary contribution scheme $a_i^j : E_i \rightarrow Y_i^j$.³

In vertical contracts, many upstream firms (for example, IBM and Apple, principals) compete for inputs produced by many downstream firms (for example, D-Ram from Samsung and S-Ram from NEC, agents). The i th component y_i^j in an action y^j is a monetary payment by an upstream firm j (a principal) to a downstream firm i (an agent). Each downstream firm i supplies an input to each upstream firm that is necessary for production. Therefore, a typical effort e_i consists of j components: $e_i = \{e_i^1, \dots, e_i^j\} \in E_i = \times_{t=1}^J E_i^t$. e_i^j is an amount of the input supplied by downstream firm i to upstream firm j . Each upstream firm j offers a demand schedule $a_i^j : E_i^j \rightarrow Y_i^j$ to each downstream firm i .⁴

In *competing first-price auctions*, each buyer j (a principal) submits a bid function to each auctioneer i (an agent). A typical action that buyer i takes is $y_i = \{y_1^j, \dots, y_I^j\} \in Y^j = \times_{s=1}^I Y_s^j$, where y_i^j is a monetary payment from buyer j to auctioneer i . Each auctioneer i has

³Dixit, Grossman, and Helpman (1997) construct a lobbying game in a common agency framework.

⁴Many examples of vertical contracts in a single-principal-multi-agent framework can be founded in Segal (1997) and Segal and Whinston (2003)

a divisible object. A typical object e_i is therefore expressed by an array $e_i = \{e_i^1, \dots, e_i^J\} \in E_i = \times_{t=1}^J E_i^t$, where e_i^j is the fraction of the object that auctioneer i sells to buyer i . $a_i^j : E_i^j \rightarrow Y_i^j$ is a bid function submitted by buyer i to auctioneer j . Given bid functions submitted by buyers, each auctioneer allocates his divisible object to maximize the revenue.⁵

While Prat and Rustichini provide a simple game to analyze bilateral contracting, it is important that in their game, principals are only allowed to offer single incentive contracts to agents. There is no market information that agents reveal to principals since agents decide only whether or not they take incentive contracts offered by principals. It restricts a principal's ability to change his contracts in response to market changes. If a principal offers mechanisms sophisticated enough to induce agents to reveal their market information about what other principals are doing, the principal is able to punish other principals' deviations from implicit collusion by changing his contracts. Even in a static game, equilibria relative to complex Γ can support many collusive outcomes that are not supported by equilibria in the Prat and Rustichini's game.

Prat and Rustichini do not allow a random incentive contract. An incentive contract a_i^j in Prat and Rustichini is also restrictive in the sense that monetary payments to agent i depend only on the effort that agent i takes. When an incentive contract makes monetary payments to an agent contingent on all the agents' efforts ($a_i^j : E \rightarrow Y_i^j$), it creates strategic externalities between agents. This strategic externalities can be used as a device to coordinate agents' efforts even without direct externalities between agents.

3 Equilibrium in Bilateral contracting Games

This section constructs a bilateral contracting game relative to some arbitrary Γ . The communication behavior and the effort decision for each agent i depend on the realized valuation on her preferences and a collection of mechanisms that all principals offer. A

⁵Bernheim and Whinston (1986) model the single first-price auction for divisible goods under complete information. In the Bernheim and Whinston's model, multiple buyers, interpreted as principals, submit their bid functions to the single auctioneer, interpreted as a common agent.

continuation strategy m_i for agent $i \in \mathcal{I}$ is a measurable mapping from $\Gamma \times \Omega_i$ into $\Delta(\mathcal{C}^J \times E_i)$ that describes the joint probability distribution on $\mathcal{C}^J \times E_i$ that agent i will use as a function of agent i 's valuation and a collection of mechanisms offered by principals.

$$m_i : \Gamma \times \Omega_i \rightarrow \Delta(\mathcal{C}^J \times E_i)$$

where \mathcal{C}^J is the set of messages that agent i can send to all principals. With a slight abuse of notation, $m_i(\cdot, \cdot | \gamma, \omega_i)$ is the probability distribution on $\mathcal{C}^J \times E_i$ that agent i uses when $\gamma = (\gamma_1^1, \dots, \gamma_I^J) \in \Gamma$ is the collection of mechanisms that principals offer to agents and agent i 's valuation is $\omega_i \in \Omega_i$.⁶

A profile of continuation strategies $\tilde{m} = (\tilde{m}_1, \dots, \tilde{m}_I)$ is a continuation equilibrium relative to Γ if for every $i \in \mathcal{I}$, every $\gamma \in \Gamma$, and every $\omega_i \in \Omega_i$, the continuation strategy \tilde{m}_i maximizes

$$\int u(a(e), e, \omega_i) d\gamma(c) d\tilde{m}_{-i}(c_{-i}, e_{-i} | \gamma, \omega_{-i}) dF(\omega_{-i} | \omega_i)$$

where $F(\cdot | \omega_i)$ is the probability distribution over the other agents' valuations conditional on agent i 's valuation ω_i , $e = (e_1, \dots, e_I)$, $a(e) = (a_1^1(e), \dots, a_I^J(e))$, and $\tilde{m}_{-i}(c_{-i}, e_{-i} | \gamma, \omega_{-i}) = \tilde{m}_1(c_1, e_1 | \gamma, \omega_1) \times \dots \times \tilde{m}_{i-1}(c_{i-1}, e_{i-1} | \gamma, \omega_{i-1}) \times \tilde{m}_{i+1}(c_{i+1}, e_{i+1} | \gamma, \omega_{i+1}) \times \dots \times \tilde{m}_I(c_I, e_I | \gamma, \omega_I)$.

Fix a continuation equilibrium \tilde{m} relative to Γ . One can derive from \tilde{m}_i the probability distribution π_i on $\mathcal{A}_i \times E_i$ for agent i given any $\gamma \in \Gamma$ and any $\omega_i \in \Omega_i$.⁷ Given the continuation equilibrium \tilde{m} relative to Γ , the probability distribution π on $\mathcal{A} \times E$ conditional

⁶One natural alternative to construct the agent's decision process is that agents simultaneously send messages to principals and then simultaneously decide efforts without observing messages sent by other agents and contracts assigned to other agents. In this case, a communication strategy h_i is a mapping from $\Gamma \times \Omega_i$ into $\Delta(\mathcal{C}^J)$. An effort strategy ϵ_i is a mapping from $\Gamma \times \Omega_i \times \mathcal{C}^J$ into $\Delta(E_i)$. For any pair of a communication strategy h_i and an effort strategy ϵ_i , there is a corresponding continuation strategy m_i such that for every $\gamma \in \Gamma$, and every $\omega_i \in \Omega_i$,

$$m_i(c_i, e_i | \gamma, \omega_i) = \epsilon_i(e_i | c_i, \gamma, \omega_i) \times h_i(c_i | \gamma, \omega_i) \quad (1)$$

where $c_i = (c_i^1, \dots, c_i^J) \in \mathcal{C}^J$ is an array of messages that agent i sends to principals. For any continuation strategy m_i , there is also a pair of a communication strategy h_i and an effort strategy ϵ_i satisfying (1).

Therefore, two approaches are strategically equivalent.

⁷The fact that π_i does depend on \tilde{m}_i is suppressed to make the notation easy.

on any $\gamma \in \Gamma$ and any $\omega = (\omega_1, \dots, \omega_I) \in \Omega$ is

$$\pi(\alpha, e|\gamma, \omega) = \pi_1(\alpha_1, e_1|\gamma, \omega_1) \times \dots \times \pi_I(\alpha_I, e_I|\gamma, \omega_I)$$

where $\alpha = (\alpha_1^1, \dots, \alpha_I^J) \in \mathcal{A}$ and $\alpha_i = (\alpha_i^1, \dots, \alpha_i^J) \in \mathcal{A}_i$ for all $i \in \mathcal{I}$.

Given any $\gamma \in \Gamma$ and any $\omega_i \in \Omega_i$, agent i 's equilibrium payoff is

$$U(\tilde{m}, \gamma, \omega_i) = \tag{2}$$

$$\int u(a(e), e, \omega_i) d\alpha(a) d\pi(\alpha, e|\gamma, \omega) dF(\omega_{-i}|\omega_i)$$

Principal $j \in \mathcal{J}$ chooses a strategy σ^j from $\Delta(\Gamma^j)$. $\sigma^j(\gamma^j)$ is the probability that principal j offers the array of mechanisms $\gamma^j = (\gamma_1^j, \dots, \gamma_I^j) \in \Gamma^j$. Suppose that the other principals' strategies are

$$\sigma^{-j}(\gamma^{-j}) = \{\sigma^1(\gamma^1), \dots, \sigma^{j-1}(\gamma^{j-1}), \sigma^{j+1}(\gamma^{j+1}), \dots, \sigma^J(\gamma^J)\}$$

Principal j 's payoff associated with a strategy σ^j is

$$V^j(\sigma^j, \sigma^{-j}, \tilde{m}) =$$

$$\int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e|\gamma, \omega) dF(\omega) d\sigma^j(\gamma^j) d\sigma^{-j}(\gamma^{-j})$$

$(\tilde{\sigma}, \tilde{m})$ is a *Perfect Bayesian Equilibrium* (PBE) relative to Γ such that $\tilde{\sigma} = (\tilde{\sigma}^1, \dots, \tilde{\sigma}^J)$ is a Nash equilibrium for the normal form game defined by the continuation equilibrium \tilde{m} relative to Γ .

4 Menus and Nature of Competition

A menu can be thought of as a simple mechanism with the message space equal to a set of incentive contracts. Among alternatives of incentive contracts, an agent simply picks up some incentive contract that she likes. A menu $\bar{\gamma}_i^j$ is a measurable mapping from \mathcal{A}_i^j into \mathcal{A}_i^j satisfying

$$\bar{\gamma}_i^j(\alpha_i^j) = \begin{cases} \alpha_i^j & \alpha_i^j \in Z \\ \tilde{\alpha}_i^j & \alpha_i^j \notin Z \end{cases}$$

for some closed subset $Z \subset \mathcal{A}_i^j$, where $\bar{\gamma}_i^j(\alpha_i^j)$ is the incentive contract that principal j assigns when agent i chooses α_i^j and $\tilde{\alpha}_i^j$ is an arbitrary incentive contract in Z . Let $\bar{\Gamma}_i^j$ be the set of all menus that principal j can offer to agent i . $\bar{\Gamma}^j = \times_{s=1}^I \bar{\Gamma}_s^j$ is the set of menus that principal j can offer to all the agents. $\bar{\Gamma} = \times_{t=1}^J \bar{\Gamma}^t$ is the set of collections of menus that all principals can offer to all agents. Let (\hat{v}, \hat{q}) be a PBE relative to $\bar{\Gamma}$. Theorem 1 shows that equilibrium payoffs for principals and agents associated with pure strategy equilibria relative to any Γ are preserved as equilibrium payoffs by pure strategy equilibria relative to $\bar{\Gamma}$.⁸

Theorem 1 *Let $(\tilde{\gamma}, \tilde{m})$ be a pure strategy PBE relative to any Γ . Then, there exists a pure strategy PBE $(\hat{\gamma}, \hat{q})$ relative to $\bar{\Gamma}$ such that*

1. $\forall i \in \mathcal{I}, \forall \omega_i \in \Omega,$

$$U(\tilde{m}, \tilde{\gamma}, \omega_i) = U(\hat{q}, \hat{\gamma}, \omega_i)$$

2. $\forall j \in \mathcal{J},$

$$V^j(\tilde{\gamma}^j, \tilde{\gamma}^{-j}, \tilde{m}) = V^j(\hat{\gamma}^j, \hat{\gamma}^{-j}, \hat{q})$$

Proof. See Appendix 9.1.

It is worthwhile to compare theorem 1 with the menu theorems for common agency. Menu theorems for common agency show that equilibria relative to $\bar{\Gamma}$ preserve equilibrium payoffs for principals and agents associated with all equilibria relative to any Γ , but theorem 1 shows that in bilateral contracting, pure strategy equilibria relative to $\bar{\Gamma}$ preserve equilibrium payoffs for principals and agents associated with pure strategy equilibria relative to any Γ . The key point of theorem 1 is that we only need to convert some communication

⁸If Γ is smaller than $\bar{\Gamma}$, then it might be possible that some pure strategy equilibria relative to Γ cannot be reproduced by any of equilibria relative to $\bar{\Gamma}$. However, it implies that those equilibria are no longer equilibria once principals are allowed to use more complex mechanisms such as menus. Those equilibria are not interesting because at least conceptually principals can use any mechanisms they like. Section 1.5 shows that any equilibria, including mixed strategy equilibria, relative to $\bar{\Gamma}$ are weakly robust in the sense that they are still equilibria even if principals are allowed to use more complex mechanisms.

and effort decisions for agents off the equilibrium path in the original game into agents' decisions off the equilibrium path in the game relative to $\bar{\Gamma}$ in a way that any unilateral deviation by a principal in $\bar{\Gamma}$ is unprofitable.

Fix a pure strategy PBE $(\tilde{\gamma}, \tilde{m})$ relative to any Γ . First, a mapping $G : \bar{\Gamma}_i^j \rightarrow \Gamma_i^j$ for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$ is constructed as follows. Consider an arbitrary menu $\bar{\gamma}_i^j$. If $\bar{\gamma}_i^j(\mathcal{A}_i^j)$ is the same as the image of the equilibrium mechanism $\tilde{\gamma}_i^j(\mathcal{C}_i^j)$, $G(\bar{\gamma}_i^j) = \tilde{\gamma}_i^j$. Otherwise, $G(\bar{\gamma}_i^j)$ is an arbitrary mechanism γ_i^j satisfying $\gamma_i^j(\mathcal{C}) = \bar{\gamma}_i^j(\mathcal{A}_i^j)$. For all $i \in \mathcal{I}$, the equilibrium probability distribution $\pi_i(\cdot, \cdot | G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \omega_i)$ on $\mathcal{A}_i \times E_i$ conditional on $(G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \omega_i)$ can be derived from \tilde{m}_i .

A continuation equilibrium $\hat{q} = (\hat{q}_1, \dots, \hat{q}_I)$ relative to $\bar{\Gamma}$ is constructed as follows. The continuation strategy \hat{q}_i for agent i satisfies

$$\hat{q}_i(\cdot, \cdot | \bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J, \omega_i) = \pi_i(\cdot, \cdot | G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \omega_i) \quad (3)$$

for every collection of menus $\bar{\gamma} = (\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J) \in \bar{\Gamma}$ and every valuation $\omega_i \in \Omega_i$.

Unlike common agency, agent i 's payoff directly depends on incentive contracts and efforts chosen by other agents because of externalities between agents. When the continuation strategy for agent i is constructed to satisfy (3), $\pi_{-i}(\alpha_{-i}, e_{-i} | G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \cdot) = \prod_{s \neq i} \pi_s(\alpha_s, e_s | G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \cdot)$ is the probability distribution over incentive contracts and efforts chosen by the other agents when the collection of menus is $(\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J) \in \bar{\Gamma}$.

In the original game, the equilibrium continuation strategy for agent i maximizes her payoff given $\pi_{-i}(\cdot, \cdot | G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \cdot)$. Since agent i can directly choose any incentive contract in the corresponding menu that she could have chosen in any mechanism, any mechanism and its corresponding menu provide the same choice set of incentive contracts. Therefore, it is optimal for agent i to use the communication strategy \hat{q}_i that satisfies (3). When each principal j offers menus $(\hat{\gamma}_1^j, \dots, \hat{\gamma}_I^j) \in \bar{\Gamma}^j$ such that for all $i \in \mathcal{I}$, the image of $\hat{\gamma}_i^j$ is equal to the image of the equilibrium mechanism $\tilde{\gamma}_i^j$ in the original game, the equilibrium probability distribution chosen by each agent i in the game relative to $\bar{\Gamma}$ is $\pi_i(\cdot, \cdot | \hat{\gamma}_1^1, \dots, \hat{\gamma}_I^J, \omega_i)$ for all $\omega_i \in \Omega_i$. The equilibrium probability distribution over payoff-relevant variables are preserved. Therefore, equilibrium payoffs for principals and agents

in the original game are preserved when principals offer the menus corresponding to the equilibrium mechanisms.

Suppose that some principal j unilaterally deviates to some menu $(\bar{\gamma}_1^j, \dots, \bar{\gamma}_I^j) \in \bar{\Gamma}^j$ in the game relative to $\bar{\Gamma}$. By the construction of the continuation equilibrium relative to $\bar{\Gamma}$, this principal will get the same payoff as the one he gets when he deviates to some mechanisms in the original game that can be converted into $(\bar{\gamma}_1^j, \dots, \bar{\gamma}_I^j)$. Therefore any deviations by each principal j in $\bar{\Gamma}^j$ do not generate higher payoffs than does the equilibrium array of menus. So, equilibrium payoffs associated with pure strategy equilibria relative to any Γ are preserved as equilibrium payoffs by pure strategy equilibria relative to $\bar{\Gamma}$.

5 Robust Equilibria

One of the concerns about specifying a set of simple mechanisms is that competition relative to the set of simple mechanisms might generate equilibria that disappear once principals are allowed to use complex mechanisms. Predictions from such equilibria might be questionable because at least conceptually principals can use any mechanisms.

Theorem 1 shows that as long as our attention is equilibrium payoffs associated with pure strategy equilibria, $\bar{\Gamma}$ is enough to reproduce equilibrium payoffs associated with pure strategy equilibria relative to any Γ . This is particularly important because pure strategy equilibria are of interest in many applications. This section asks whether the competition relative to the set of menus generates *robust equilibria* in bilateral contracting in the sense that all equilibria relative to $\bar{\Gamma}$ persist even if the set of mechanisms is enlarged.

Peters (2001) shows that in common agency, there is always a way to assign a continuation equilibrium relative to any Γ bigger than $\bar{\Gamma}$ such that equilibrium payoffs associated with equilibria relative to $\bar{\Gamma}$ are equilibria payoffs relative to any Γ . Therefore, any equilibrium relative to $\bar{\Gamma}$ is *weakly robust*. Formally, $\Gamma = \times_{i,j} \Gamma_i^j$ is bigger than $\bar{\Gamma} = \times_{i,j} \bar{\Gamma}_i^j$ ($\Gamma \succ \bar{\Gamma}$) if there exists an embedding $\eta : \bar{\Gamma} \rightarrow \Gamma$. It implies that there are more mechanisms in Γ than in $\bar{\Gamma}$. The weak robustness of equilibria relative to $\bar{\Gamma}$ in common agency also holds in bilateral contracting.

Theorem 2 *Let (\hat{v}, \hat{q}) be a PBE relative to $\bar{\Gamma}$. For every compact metric $\Gamma \succ \bar{\Gamma}$ and every PBE (\hat{v}, \hat{q}) relative to $\bar{\Gamma}$, there exists a PBE $(\tilde{\sigma}, \tilde{m})$ relative to Γ that preserves equilibrium payoffs associated with a PBE (\hat{v}, \hat{q}) relative to $\bar{\Gamma}$.*

proof. See Appendix 9.2

Consider an equilibrium relative to $\bar{\Gamma}$. A principal knows that there are no other menus that provide higher payoffs for him than do menus in the support of his equilibrium strategy. Suppose that Γ is bigger than $\bar{\Gamma}$. A principal will deviate to mechanisms in the new set if it is profitable. Suppose that for example, $(\bar{\gamma}_1^1, \bar{\gamma}_2^1, \dots, \bar{\gamma}_I^1)$ is a collection of menus that principals offer to agents. If principal 1 unilaterally deviates and offers agent 1 γ_1^1 in Γ_1^1 while the other principals still offer agent 1 menus, $(\gamma_1^1, \bar{\gamma}_2^1, \dots, \bar{\gamma}_I^1)$ is the collection of mechanisms. Agent 1 might choose different incentive contracts in the mechanisms $(\gamma_1^1, \bar{\gamma}_2^1, \dots, \bar{\gamma}_I^1)$ offered to her and subsequently a different effort. The reason is that γ_1^1 might make some other agents believe that agent 1 would choose different incentive contracts and a different effort. So, these agents might change incentive contracts and efforts that they choose. Even if the mechanism γ_1^1 offers agent 1 the same menu of alternatives as $\bar{\gamma}_1^1$, it may trigger a new continuation equilibrium in which agent 1 is worse off while the principal is better off. This possibility does not arise in common agency. In bilateral contracting, this is an important reason why some principal might like to deviate to complex mechanisms in order to coordinate agents' behavior in his interest.

Any mechanism γ_i^j in Γ_i^j has the corresponding menu $\bar{\gamma}_i^j$ in $\bar{\Gamma}_i^j$. The choice sets of incentive contracts that γ_i^j and $\bar{\gamma}_i^j$ provide are exactly the same. It follows that each agent's equilibrium communication and effort decision at any collection of complex mechanisms can be constructed to choose the same incentive contracts and the same effort that she could have chosen at a collection of corresponding menus in the continuation equilibrium relative to $\bar{\Gamma}$. If a principal deviates to mechanisms that can be converted into menus in the support of his equilibrium strategy relative to $\bar{\Gamma}$, his payoff is the same as the equilibrium payoff relative to $\bar{\Gamma}$. If he deviates to mechanisms that can be converted into menus outside of the support of his equilibrium strategy, his payoff is no higher than the equilibrium payoff

relative to $\bar{\Gamma}$.

From theorem 1 and theorem 2, we can conclude that the set of equilibrium payoffs associated with pure strategy equilibria relative to $\bar{\Gamma}$ is the same as the set of equilibrium payoffs associated with pure strategy equilibria relative to any $\Gamma \succcurlyeq \bar{\Gamma}$ and that the set of equilibria payoffs associated with all equilibria relative to $\bar{\Gamma}$ is a subset of the set of equilibrium payoffs associated with all equilibria relative to any $\Gamma \succcurlyeq \bar{\Gamma}$.

6 Mixed Strategy Equilibria

Theorem 1 shows that equilibrium payoffs associated with pure strategy equilibria relative to any Γ can be preserved by pure strategy equilibria relative to $\bar{\Gamma}$. It still remains to answer the question of whether equilibrium payoffs associated with mixed strategy equilibria relative to any Γ can be preserved by equilibria relative to $\bar{\Gamma}$. This question is directly related to the externalities between agents and the complexity of mechanisms in Γ .

With externalities between agents, optimal incentive contracts and efforts for one agent depend on incentive contracts and efforts that other agents choose. It is therefore natural to expect different continuation equilibria, which generate different payoffs for principals, at different collections of mechanisms in Γ even if these different collections are converted into one collection of menus $(\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J)$. Principals can correlate these different continuation equilibria by randomizing their mechanisms. If principals are restricted to use menus, only one continuation equilibrium is assigned at the collection of menus $(\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J)$. Therefore such a coordination is not possible. This type of problems does not arise in common agency because there is only one agent in the model. Example 3 provides a mixed strategy equilibrium relative to some Γ whose payoffs cannot be preserved by equilibria relative to $\bar{\Gamma}$.

Example 3 *There are two principals and two agents with complete information and agents take no effort. The set of actions is $Y^1 = Y^2 = \{(a, a), (a, b), (b, a), (b, b)\}$, where the first component in each action is for agent 1 and the second component for agent 2. The payoffs for each player are given by*

	a, a	a, b	b, a	b, b
a, a	0, 0, 10, 10	4, 2, 4, 4	0, 0, 10, 10	4, 1, 2, 4
a, b	0, 0, 10, 10	5, 6, 4, 3	0, 0, 10, 10	7, 3, 4, 4
b, a	0, 0, 10, 10	0, 0, 10, 10	0, 0, 10, 10	0, 0, 10, 10
b, b	0, 0, 10, 10	0, 0, 10, 10	0, 0, 10, 10	0, 0, 10, 10

Table 1: Payoffs

Each cell in the first row coincides with an action chosen by principal 2: the first component for agent 1 and the second for agent 2. Each cell in the first column is an action chosen by principal 1: the first component for agent 1 and the second for agent 2. The numbers in the cells give the payoffs for principal 1, principal 2, agent 1, and agent 2 respectively.

Suppose that principals are restricted to offer menus to agents. There are three menus in this example: $m_a = \{a\}$, $m_b = \{b\}$, and $m = \{a, b\}$ that each principal can offer to each agent. Each principal offers menus, one for each agent. Each agent chooses components from menus offered to her. Table 2 summarizes subgames played by agents given arrays of menus.

Each cell in the first row coincides with a pair of menus offered by principal 2: the first menu to agent 1 and the second to agent 2. Each cell in the first column coincides with a pair of menus offered by principal 1: the first menu to agent 1 and the second to agent 2. Suppose that principal 1 offers (m_a, m) and principal 2 offers (m, m_b) . Agent 1 must choose the component a in the principal 1's menu m_a because there is no other component that she can choose in m_a . But, she can choose either a or b in the menu m offered by principal 2. Agent 2 can choose either a or b in the menu m offered by principal 1 while b is the only choice in the menu m_b offered by principal 2. Agents play the 2×2 normal form game described in the corresponding cell in table 2. There are only two equilibria in this game. One is that agent 1 chooses a in both principals' menus m_a and m respectively and agent 2 chooses a in the principal 1's menu m and b in the principal 2's menu m_b . The

	m_a, m_b		m_b, m_b		m, m_b		\dots		
m_a, m_a	4,2,4,4		4,1,2,4		a,b	a,a 4,2,4,4	0,0,10,10		
					a,b	3,1,2,4			
m_a, m_b	5,6,4,3		7,3,4,4		b,b	a,a 5,6,4,3	0,0,10,10		
					a,b	7,3,4,4			
m_a, m	a,b	b,b	a,b	b,b	a,b	a,a 4,2,4,4	5,6,4,3	0,0,10,10	
	a,a	4,2,4,4	5,6,4,3	a,b	4,1,2,4	7,3,4,4	a,b	4,1,2,4	7,3,4,4
\vdots	0,0,10,10		0,0,10,10		0,0,10,10		0,0,10,10		

Table 2: Subgames played by agents

other is that agent 1 chooses a in the principal 1's menu m_a and b in the principal 2's menu m and agent 2 chooses b in both principals' menus m and m_b respectively.

Suppose that principal 1 offers (m_a, m) and principal 2 offers (m_b, m_b) . Agent 1 must choose a in the principal 1's menu m_a and b in the principal 2's menu m_b . Agent 2 chooses either a or b in the principal 1's menu m but only b in the principal 2's menu m_b . In this case, agent 2 is indifferent between a and b from principal 1. So, it is optimal for agent 2 to choose a with probability p and b with probability $1 - p$ from principal 1 for any $p \in [0, 1]$.

Consider the case where principal 1 offers (m_a, m_b) and principal 2 offers (m, m_b) . Agent 2 chooses b in both principals' menus m_b and m_b respectively. Agent 1 can choose either a or b in the principal 2's menu m but only a in the principal 1's menu m_a . Since agent 1 is indifferent between two components in the principal 2's menu m in this case, it is optimal for agent 1 to choose a with probability r and b with probability $1 - r$ in the principal 2's menu m , where $r \in [0, 1]$.

Given strategies p and r , principals play a normal form game given by table 3, where $p \in [0, 1]$, $r \in [0, 1]$, $z \in \{0, 1\}$.

	m_a, m_b	m_b, m_b	m, m_b	\dots
m_a, m_a	4, 2	4, 1	4, 2	0, 0
m_a, m_b	5, 6	7, 3	$7 - 2r, 3 + 3r$	0, 0
m_a, m	4, 2	$7 - 3p, 3 - 2p$	$7 - 3z, 3 - z$	0, 0
\vdots	0, 0	0, 0	0, 0	0, 0

Table 3: A continuation equilibrium relative to the set of menus

Consider a bilateral contracting game relative to some $\Gamma = \Gamma_1^1 \times \Gamma_2^1 \times \Gamma_1^2 \times \Gamma_2^2$. Suppose that $\Gamma_i^j = \{\gamma_a, \gamma_b, \gamma, \gamma'\}$ for all $j \in \{1, 2\}$ and all $i \in \{1, 2\}$. For simplicity $\mathcal{C} = M$. In particular, the image of mechanisms satisfies $\gamma_a(M) = \{a\}$, $\gamma_b(M) = \{b\}$, and $\gamma(M) = \gamma'(M) = \{a, b\}$. Note that essentially γ and γ' provide the same choice set of alternative components although each mechanism can require different messages to assign the same component.

One possible normal form game defined by a continuation equilibrium relative to Γ is

	γ_a, γ_b	γ_b, γ_b	γ', γ_b	γ, γ_b	\dots
γ_a, γ_a	4, 2	4, 1	4, 2	4, 2	0, 0
γ_a, γ_b	5, 6	7, 3	5, 6	5, 6	0, 0
γ_a, γ	4, 2	4, 1	7, 3	4, 2	0, 0
γ_a, γ'	4, 2	4, 1	4, 2	7, 3	0, 0
\vdots	0, 0	0, 0	0, 0	0, 0	0, 0

Table 4: A continuation equilibrium relative to Γ

A mixed strategy equilibrium in the normal form game is that principal 1 offers (γ_a, γ) or (γ_a, γ') with equal probability and principal 2 offers (γ, γ_b) or (γ', γ_b) with equal probability. In this equilibrium, principal 1 gets 5.5 and principal 2 gets 2.5. Both agents always get 4. These equilibrium payoffs are never supported by any equilibria relative to $\bar{\Gamma}$. Two pairs of mechanisms (γ', γ_b) and (γ, γ_b) offered by principal 2 are converted into the pair of menus (m, m_b) . Two pairs of mechanisms (γ_a, γ) and (γ_a, γ') offered by principal 1 are converted

into the pair of menus (m_a, m) . If we restrict principals to use menus, then either only $(7, 3)$ or only $(4, 2)$ is assigned as payoffs for principals when principal 1 offers (m_a, m) and principal 2 offers (m, m_b) . Principals cannot make their actions correlated in a way that they do in the equilibrium relative to Γ . In general, menus restrict a principal's ability to correlate his actions across agents in his interest.

7 Correlated Equilibria relative to Menus

Example 3 highlights the reason why equilibrium payoffs associated with some mixed strategy equilibria relative to some Γ cannot be preserved by equilibria relative to $\bar{\Gamma}$. Consider different collections of mechanisms in Γ that can be converted into one collection of menus in $\bar{\Gamma}$. Since there are externalities between agents, it is natural to expect that agents act differently at different collections of mechanisms and induce different continuation equilibria, which generate different payoffs for principals. Essentially, names of mechanisms coordinate agents' choices over incentive contracts. In other words, they deliver agents a belief about which equilibrium will occur and agents' optimal behavior confirms the belief in equilibrium.

This type of equilibria is closely related to *correlated equilibria*. This section considers correlated equilibria relative to $\bar{\Gamma}$. There is a mapping ζ from $\bar{\Gamma}$ into the set of probability distributions over the set of states S .

$$\zeta : \bar{\Gamma} \rightarrow \Delta(S)$$

A mapping ζ is called a random state mapping. With a slight abuse of notation, $\zeta(\cdot | \bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J)$ is a probability distribution on S when $(\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J)$ is a collection of menus offered by all the principals. For each $i \in \mathcal{I}$, an information partition mapping \mathcal{P}_i is a mapping from $\bar{\Gamma} \times \Omega_i$ into Ξ , where Ξ is the set of all feasible partitions of S . $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_I)$ is called a profile of information partition mappings.

First, principals simultaneously offer menus to agents. A collection of menus determines a probability distribution on S . A state is realized from the probability distribution. After

seeing a collection of menus and a state, each agent simultaneously chooses an effort and incentive contracts from menus that principals offer to her.

A continuation strategy g_i for agent $i \in \mathcal{I}$ is a measurable mapping from $\bar{\Gamma} \times S \times \Omega_i$ into $\Delta(\mathcal{A}_i \times E_i)$ that describes the joint probability distribution over $\mathcal{A}_i \times E_i$ that agent i will use as a function of a collection of menus, a state, and agent i 's valuation.

$(\hat{\zeta}, S, \hat{\mathcal{P}}, \hat{g})$ is a continuation equilibrium relative to $\bar{\Gamma}$ if (a) $\hat{g}_i(\cdot|\bar{\gamma}, s, \omega_i) = \hat{g}_i(\cdot|\bar{\gamma}, s', \omega_i)$ whenever $s \in P_i$ and $s' \in P_i$ for some $P_i \in \mathcal{P}_i(\bar{\gamma}, \omega_i)$ for each $i \in \mathcal{I}$ given any $\bar{\gamma} = (\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J) \in \bar{\Gamma}$ and any $\omega_i \in \Omega_i$ and (b) for every $i \in \mathcal{I}$, every $s \in S$, every $\bar{\gamma} = (\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J) \in \bar{\Gamma}$, and every $\omega_i \in \Omega_i$, the communication strategy \hat{g}_i maximizes

$$U(\hat{g}, \bar{\gamma}, s, \omega_i) = \int u(a(e), e, \omega_i) d\alpha(a) d\hat{g}_{-i}(\alpha_{-i}, e_{-i}|\bar{\gamma}, s, \omega_{-i}) dF(\omega_{-i}|\omega_i)$$

where $\hat{g}_{-i}(\alpha_{-i}, e_{-i}|\bar{\gamma}, s, \omega_{-i}) = \hat{g}_1(\alpha_1, e_1|\bar{\gamma}, s, \omega_1) \times \dots \times \hat{g}_{i-1}(\alpha_{i-1}, e_{i-1}|\bar{\gamma}, s, \omega_{i-1}) \times \hat{g}_{i+1}(\alpha_{i+1}, e_{i+1}|\bar{\gamma}, s, \omega_{i+1}) \times \dots \times \hat{g}_I(\alpha_I, e_I|\bar{\gamma}, s, \omega_I)$.

Principal $j \in \mathcal{J}$ chooses a strategy ν^j from $\Delta(\bar{\Gamma}^j)$. $\nu^j(\bar{\gamma}^j)$ is the probability that principal j offers the array of menus $\bar{\gamma}^j = (\bar{\gamma}_1^j, \dots, \bar{\gamma}_I^j) \in \bar{\Gamma}^j$. Suppose that the strategies chosen by the other principals are $\nu^{-j}(\bar{\gamma}^{-j}) = (\nu^1(\bar{\gamma}^1), \dots, \nu^{j-1}(\bar{\gamma}^{j-1}), \nu^{j+1}(\bar{\gamma}^{j+1}), \dots, \nu^J(\bar{\gamma}^J))$. Principal j 's payoff associated with a strategy ν^j is

$$V^j(\nu^j, \nu^{-j}, \hat{g}) =$$

$$\int v^j(a(e), e, \omega) d\alpha(a) \hat{g}(\alpha, e|\bar{\gamma}, s, \omega) d\zeta(s|\bar{\gamma}) d\nu^j(\bar{\gamma}^j) d\nu^{-j}(\bar{\gamma}^{-j}) dF(\omega)$$

where $\hat{g}(\alpha, e|\bar{\gamma}, s, \omega) = \hat{g}(\alpha_1, e_1|\bar{\gamma}, s, \omega_1) \times \dots \times \hat{g}(\alpha_I, e_I|\bar{\gamma}, s, \omega_I)$ with $\alpha = (\alpha_1, \dots, \alpha_I)$, $e = (e_1, \dots, e_I)$, and $\omega = (\omega_1, \dots, \omega_I)$. $(\hat{\zeta}, S, \hat{\mathcal{P}}, \hat{g}, \hat{\nu})$ is a *Correlated Equilibrium* (CE) relative to $\bar{\Gamma}$ such that $\hat{\nu} = (\hat{\nu}^1, \dots, \hat{\nu}^J)$ is a Nash equilibrium for the normal form game defined by the continuation equilibrium $(\hat{\zeta}, S, \hat{\mathcal{P}}, \hat{g})$ relative to $\bar{\Gamma}$.

The key question is how big S should be in order to preserve equilibrium payoffs associated with all equilibria relative to any Γ as equilibrium payoffs associated with correlated equilibria relative to $\bar{\Gamma}$. Theorem 4 shows that if $S = \Delta(Y \times E \times \Omega)$, equilibrium payoffs associated with all equilibria relative to any Γ can be preserved by correlated equilibria relative to $\bar{\Gamma}$.

Theorem 4 *Let $(\tilde{\sigma}, \tilde{m})$ be a PBE relative to any Γ . Suppose that $S = \Delta(Y \times E \times \Omega)$. Then, there exists a CE $(\hat{\zeta}, S, \hat{\mathcal{P}}, \hat{g}, \hat{\nu})$ relative to $\bar{\Gamma}$ satisfying*

1. $\forall i \in \mathcal{I}, \forall \gamma \in \times_{t=1}^J \text{supp } \tilde{\sigma}^t, \forall \omega_i \in \Omega_i$, there exist $\bar{\gamma} \in \times_{t=1}^J \text{supp } \hat{\nu}^t$ and $s \in \text{supp } \hat{\zeta}(\bar{\gamma})$ such that

$$U(\tilde{m}, \tilde{\gamma}, \omega_i) = U(\hat{g}, \bar{\gamma}, s, \omega_i)$$

2. $\forall j \in \mathcal{J}$

$$V^j(\tilde{\sigma}^j, \tilde{\sigma}^{-j}, \tilde{m}) = V^j(\hat{\nu}^j, \hat{\nu}^{-j}, \hat{g})$$

Proof. See Appendix 9.3.

Theorem 4 shows that for any equilibrium relative to any Γ , there exists a pair of a random state mapping and a profile of equilibrium strategies such that equilibrium payoffs associated with any equilibria relative to any Γ can be preserved as equilibrium payoffs associated with correlated equilibria relative to $\bar{\Gamma}$. It implies that there is no additional equilibrium allocation that one can learn by modelling competition between principals relative to complex Γ instead of modelling it relative to $\bar{\Gamma}$.

The key point of theorem 4 is how to choose S and to construct $\hat{\zeta}$. For any collection of menus $\bar{\gamma} \in \bar{\Gamma}$, there exist many collections of mechanisms in Γ that can be converted into $\bar{\gamma}$. For all $\bar{\gamma} = (\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J) \in \bar{\Gamma}$, let $\xi(\bar{\gamma}) = \times_{k \in \mathcal{I}, t \in \mathcal{J}} \xi_s^t(\bar{\gamma}_s^t)$ be the set of such collections of mechanisms, where $\xi_s^t(\bar{\gamma}_s^t)$ is the set of mechanisms in Γ_s^t that can be converted into $\bar{\gamma}_s^t$. Different collections of mechanisms in $\xi(\bar{\gamma})$ may induce different equilibrium allocations because of externalities between agents. It is however important that equilibrium allocations, which decide equilibrium payoffs, are fully characterized by a probability distribution on $Y \times E \times \Omega$ at any collection of mechanisms. Therefore, S can be as small as $\Delta(Y \times E \times \Omega)$ in the sense that there is a unique probability distribution on $Y \times E \times \Omega$ corresponding to any equilibrium allocation at any collection of mechanisms in $\xi(\bar{\gamma})$. On the equilibrium path, a probability distribution on $\Delta(Y \times E \times \Omega)$ conditional on $\xi(\bar{\gamma})$ can be derived from equilibrium strategies $\tilde{\sigma}$ used by principals in the original game. This is $\hat{\zeta}(\cdot | \bar{\gamma}) = \tilde{\sigma}(\cdot | \xi(\bar{\gamma})) \in \Delta(S)$. The strategies $\hat{\nu}$ for principals are induced from $\tilde{\sigma}$ used by principals in the original game.

An equilibrium allocation that occurs with a positive probability in the original game is realized as the form of the corresponding state with a positive probability in the new game. After seeing a collection of menus and a state, each agent i has the same belief about which equilibrium will occur. In the new game, agents use probability distributions over incentive contracts and efforts induced by their equilibrium continuation strategies in the original game that generate the realized state, that is a probability distribution over $Y \times E \times \Omega$. When all agents choose probability distributions in this way, each agent finds her probability distribution optimal given other agents' probability distributions. It also ensures that equilibrium payoffs for agents in the original game are preserved as equilibrium payoffs in the new game. Since $\widehat{\zeta}(\cdot|\bar{\gamma})$ for each $\bar{\gamma} \in \bar{\Gamma}$ on the equilibrium path and $\widehat{\nu}$ are induced by the equilibrium strategies that principals use in the original game, equilibrium payoffs for principals in the new game are also the same as the ones on the equilibrium path in the original game.

Suppose that principal j unilaterally deviates to menus $\bar{\gamma}^j = (\bar{\gamma}_1^j, \dots, \bar{\gamma}_I^j)$ outside of the support of $\widehat{\nu}^j$. For any collection of menus $(\bar{\gamma}^j, \bar{\gamma}^{-j}) \in \bar{\Gamma}$, there are many collections of mechanisms in Γ that can be converted into $(\bar{\gamma}^j, \bar{\gamma}^{-j})$. Choose an arbitrary mechanisms, say $\bar{\xi}^j(\bar{\gamma}^j)$, that can be converted into $\bar{\gamma}^j$. A set $\{\bar{\xi}^j(\bar{\gamma}^j)\} \times \xi^{-j}(\bar{\gamma}^{-j})$ includes collections of mechanisms that can be converted into $(\bar{\gamma}^j, \bar{\gamma}^{-j})$. Then, the probability distribution $\widehat{\zeta}(\cdot|\bar{\gamma}^j, \bar{\gamma}^{-j})$ is $\tilde{\sigma}^{-j}(\cdot|\{\bar{\xi}^j(\bar{\gamma}^j)\} \times \xi^{-j}(\bar{\gamma}^{-j}))$ on $S = \Delta(Y \times E \times \Omega)$ conditional on $\{\bar{\xi}^j(\bar{\gamma}^j)\} \times \xi^{-j}(\bar{\gamma}^{-j})$ is derived from strategies $\tilde{\sigma}^{-j}$ used by all the principals except for principal j . The construction of $\widehat{\zeta}(\cdot|\bar{\gamma}^j, \bar{\gamma}^{-j})$ ensures that the payoff for principal j associated with deviating to $\bar{\gamma}^j$ is the same as the one that he can get by deviating to $\bar{\xi}^j(\bar{\gamma}^j)$ in the original game. Since any menus in the support of $\widehat{\nu}^j$ generates the same payoff as the equilibrium payoff in the original game, any deviation outside of the support of $\widehat{\nu}^j$ is not profitable to principal j .

It is also straightforward to show that correlated equilibria relative to $\bar{\Gamma}$ are weakly robust. Suppose that principal j unilaterally deviates to some mechanisms $\gamma^j = (\gamma_1^j, \dots, \gamma_I^j)$ in $\Gamma^j \not\subseteq \bar{\Gamma}^j$. It should be noted that a probability distribution over states is conditional only on a collection of choice sets of incentive contracts that a collection of mechanisms provide,

which is a collection of corresponding menus. If γ^j can be converted into mechanisms in the support of $\widehat{\mathcal{D}}^j$, then this deviation provides the same payoff as the equilibrium payoff generated by $\widehat{\mathcal{D}}^j$. If not, then this deviation generates the same payoff as the one that principal j can get by deviating to some array of menus. Therefore, any deviation to complex mechanisms.

Equilibrium payoffs associated with the mixed strategy equilibrium relative to Γ in example 3 can be easily preserved as equilibrium payoffs associated with a correlated equilibrium relative to $\bar{\Gamma}$. Example 5 shows it.

Example 5 *The mixed strategy equilibrium described in example 3 is as follows. Principal 1 offers (γ_a, γ) or (γ_a, γ') with equal probability and principal 2 offers (γ, γ_b) or (γ', γ_b) with equal probability. When a collection of mechanisms is $((\gamma_a, \gamma), (\gamma', \gamma_b))$ or $((\gamma_a, \gamma'), (\gamma, \gamma_b))$, agent 1 chooses (a, a) and agent 2 chooses (a, b) . When a collection of mechanisms is $((\gamma_a, \gamma), (\gamma, \gamma_b))$ or $((\gamma_a, \gamma'), (\gamma', \gamma_b))$, agent 1 chooses (a, a) and agent 2 chooses (a, b) .*

Let $y = (y_1^1, y_1^2, y_2^1, y_2^2)$ be a collection of actions in the example. On the equilibrium path, (a, a, a, b) and (a, b, b, b) occur with equal probability respectively. Two pairs of mechanisms in the support of principal 1's strategy are converted into (m_a, m) . Two pairs of mechanisms in the support of principal 2's strategy are converted into (m, m_b) . Let $\widehat{\zeta}(\cdot | \bar{\gamma}_1^1, \bar{\gamma}_1^2, \bar{\gamma}_2^1, \bar{\gamma}_2^2)$ be the probability distribution over S , where $(\bar{\gamma}_1^1, \bar{\gamma}_1^2, \bar{\gamma}_2^1, \bar{\gamma}_2^2)$ is a collection of mechanisms that principals offer agents. s_1 denotes the probability distribution such that $s_1(a, a, a, b) = 1$. Let s_2 be the probability distribution such that $s_2(a, b, b, b) = 1$. $\widehat{\zeta}(\cdot | m_a, m, m, m_b)$ satisfies

$$\widehat{\zeta}(s | m_a, m, m, m_b) = \begin{cases} 1/2 & \text{if } s = s_1 \\ 1/2 & \text{if } s = s_2 \end{cases}$$

If principal 1 offers (m_a, m) and principal 2 offers (m, m_b) , s_1 and s_2 are realized with equal probability respectively. Agent 1 chooses (a, a) and agent 2 chooses (a, b) if the realized state is s_1 . Agent 1 chooses (a, b) and agent 2 chooses (b, b) if the realized state is s_2 . It is immediate that each agent's choice is a best response given the other's choice in each state.

Furthermore, equilibrium payoffs for principals and agents are reproduced when principal 1 offers (m_a, m) , principal 2 offers (m, m_b) , and each agent chooses components described as above.

Suppose that principal 1 unilaterally deviates to (m_a, m_b) . In the example, only (γ_a, γ_b) in $\Gamma^1 = \Gamma_1^1 \times \Gamma_2^1$ is converted into (m_a, m_a) . Consider the case where principal 1 unilaterally deviates to (γ_a, γ_b) in the original game while principal 2 still offers (γ, γ_b) or (γ', γ_b) respectively with equal probability. Two collections of mechanisms $(\gamma_a, \gamma_b, \gamma, \gamma_b)$ and $(\gamma_a, \gamma_b, \gamma', \gamma_b)$ are realized respectively with equal probability in the original game. It is easy to show how to construct the probability distribution over states conditional on principal 1's deviation to (m_a, m_b) in the new game. Let s'_1 be the equilibrium probability distribution when $(\gamma_a, \gamma_b, \gamma, \gamma_b)$ is the collection of mechanisms that principals offer in the original game. s'_2 denotes the equilibrium probability distribution when $(\gamma_a, \gamma_b, \gamma', \gamma_b)$ is the collection of mechanisms that principals offer in the original game. $\widehat{\zeta}(\cdot | m_a, m_b, m, m_b)$ satisfies that $\widehat{\zeta}(s'_1 | m_a, m_b, m, m_b) = 1/2$ and $\widehat{\zeta}(s'_2 | m_a, m_b, m, m_b) = 1/2$. Since $s'_1 = s'_2 = s'$ such that $s'(a, a, b, b) = 1$ in this example, $\widehat{\zeta}(\cdot | m_a, m_b, m, m_b)$ is the degenerated probability distribution such that $\widehat{\zeta}(s' | m_a, m_b, m, m_b) = 1$. When (m_a, m_b, m, m_b) is the collection of menus that principals offer and the state is s' , it is an equilibrium that agent 1 chooses (a, a) and agent 2 chooses (b, b) . Since (γ_a, γ_b) is not profitable for principal 1 in the original game, the deviation to (m_a, m_b) is not profitable in the new game as well. One can construct the probability distributions over states and agents' equilibrium choices on components in any unilateral or multilateral deviations.

8 Discussion

Menus are simple in the sense that there is no communication between players. After seeing a collection of menus (and a state), agents choose incentive contracts directly from menus without communication. Correlated equilibria relative to $\bar{\Gamma}$ uncover how competition relative to complex Γ enables principals to coordinate agents' equilibrium behavior. The different collections of mechanisms that provide the same collection of choice sets of

incentive contracts essentially decide a state that delivers agents a belief of which equilibrium will occur. Given a state, agents optimally choose their incentive contracts by sending messages and the agents' optimal behavior confirms the belief in equilibrium. Since an equilibrium allocation is fully characterized by a probability distribution over payoff-relevant variables at any collection of mechanisms, $S = \Delta(Y \times E \times \Omega)$ is big enough to associate any equilibrium allocation with a unique state. Given $S = \Delta(Y \times E \times \Omega)$, one can always find a correlated equilibrium relative to $\bar{\Gamma}$ such that it reproduces equilibrium payoffs for principals and agents associated with any equilibrium relative to any complex Γ .

It is important that the set of menus can be applied to only bilateral contracting. In collective contracting, incentive contracts are jointly determined by messages sent by many different agents, so a menu does not provide a way to reduce the complexity of arbitrary mechanisms. It is a challenging but interesting question whether there is a simple set of mechanisms in collective contracting that generates interesting collusive equilibrium allocations.

9 Appendix

We start with some basic definitions. For any subset $Z \subset \mathcal{A}_i^j$, define the mapping $\tau(Z) : \mathcal{A}_i^j \rightarrow \mathcal{A}_i^j$ such that

$$\tau(Z)(\alpha_i^j) = \begin{cases} \alpha_i^j & \alpha_i^j \in Z \\ \tilde{\alpha}_i^j & \alpha_i^j \notin Z \end{cases}$$

where $\tau(Z)(\alpha_i^j)$ is the incentive contract assigned in $\tau(Z)$ when agent i chooses α_i^j and $\tilde{\alpha}_i^j$ is an arbitrary element of Z . Let $\gamma_i^j(\mathcal{C})$ be the image of γ_i^j . Consider the map, $\psi_i^j : \Gamma_i^j \rightarrow \bar{\Gamma}_i^j$ satisfying

$$\psi_i^j : \gamma_i^j \mapsto \bar{\gamma}_i^j(\cdot) = \tau(\gamma_i^j(\mathcal{C}))(\cdot)$$

This map converts a mechanism γ_i^j into the menu of alternatives that γ_i^j provides. It is possible that two or more mechanisms provide the same menu of alternatives, so ψ_i^j is a many-to-one mapping with the inverse correspondence ξ_i^j . ψ^j is the mapping satisfying

$$\psi^j : (\gamma_1^j, \dots, \gamma_I^j) \mapsto (\bar{\gamma}_1^j(\cdot), \dots, \bar{\gamma}_I^j(\cdot))$$

where $\bar{\gamma}_i^j(\cdot) = \tau(\gamma_i^j(\mathcal{C}))(\cdot)$ for all $i \in \mathcal{I}$. Let ξ^j be the inverse correspondence of ψ^j . Finally, ψ is the mapping satisfying

$$\psi : (\gamma_1^1, \dots, \gamma_I^J) \mapsto (\bar{\gamma}_1^1(\cdot), \dots, \bar{\gamma}_I^J(\cdot))$$

where $\bar{\gamma}_i^j(\cdot) = \tau(\gamma_i^j(\mathcal{C}))(\cdot)$ for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$. Let ξ be the inverse correspondence of ψ .

Given a collection of mechanisms $\gamma \in \Gamma$ and agent i 's valuation $\omega_i \in \Omega_i$, the payoff for agent i associated with incentive contracts $\alpha_i = (\alpha_i^1, \dots, \alpha_i^J) \in \mathcal{A}_i$ and $e_i \in E_i$ is given by

$$y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot, \cdot | \gamma, \cdot)) = \int u(a(e), e, \omega_i) d\alpha_i(a_i) d\alpha_{-i}(a_{-i}) d\pi_{-i}(\alpha_{-i}, e_{-i} | \gamma, \omega_{-i}) dF(\omega_{-i} | \omega_i)$$

Since agent i 's payoffs depend on a_{-i} and e_{-i} , y does depends on $\pi_{-i}(\cdot, \cdot | \gamma, \cdot)$. The equilibrium payoff for agent i is then

$$U(\tilde{m}, \gamma, \omega_i) = \int y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot, \cdot | \gamma, \cdot)) d\pi_i(\alpha_i, e_i | \gamma, \omega_i) = \underset{\alpha_i, e_i}{Max} \{y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot, \cdot | \gamma, \cdot)) : \alpha_i^j \in \gamma_i^j(\mathcal{C}_i^j) \forall j \in \mathcal{J}, e_i \in E_i\} \quad (4)$$

The second equality holds because any array of incentive contracts and any effort that agent i chooses with positive probability must maximize her payoff.

9.1 Proof of Theorem 1

Proof. Fix a pure strategy equilibrium $\tilde{\gamma} = (\tilde{\gamma}_1^1, \dots, \tilde{\gamma}_I^J) \in \Gamma$ given a continuation equilibrium \tilde{m} relative to Γ . Given the continuation equilibrium \tilde{m} , each collection of mechanisms $\gamma = (\gamma_1^1, \dots, \gamma_I^J)$ in Γ is transformed into a collection of menus with the mapping ψ . For all $\bar{\gamma} = (\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J) \in \bar{\Gamma}$, define $(G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J))$ such that for all $j \in \mathcal{J}$ and all $i \in \mathcal{I}$

$$G(\bar{\gamma}_i^j) = \begin{cases} \tilde{\gamma}_i^j & \text{if } \tilde{\gamma}_i^j \in \xi_i^j(\bar{\gamma}_i^j) \\ \bar{\xi}_i^j(\bar{\gamma}_i^j) & \text{otherwise} \end{cases}$$

where $\bar{\xi}_i^j(\bar{\gamma}_i^j)$ is an arbitrary mechanism in $\xi_i^j(\bar{\gamma}_i^j)$. Now we can specify the continuation equilibrium relative to $\bar{\Gamma}$.

The continuation strategy for agent i is constructed as follows: for all $\bar{\gamma} = (\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J) \in \bar{\Gamma}$ and all $\omega_i \in \Omega_i$

$$\hat{q}_i(\cdot, \cdot | \bar{\gamma}, \omega_i) = \pi_i(\cdot, \cdot | G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \omega_i) \quad (5)$$

Principal j 's strategy is $(\hat{\gamma}_1^j, \dots, \hat{\gamma}_I^j) = (\psi_1^j(\bar{\gamma}_1^j), \dots, \psi_I^j(\bar{\gamma}_I^j))$.

First, we need to prove that continuation strategies described above constitute a continuation equilibrium relative to $\bar{\Gamma}$. Consider the payoff for agent i after she chooses α_i and e_i when $(\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J)$ is the collection of menus that principals offer and agent i 's valuation is ω_i . (5) implies that this payoff is

$$y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot, \cdot | \gamma, \cdot)) = \int u(a(e), e, \omega_i) d\alpha_i(a_i) d\alpha_{-i}(a_{-i}) d\pi_{-i}(\alpha_{-i}, e_{-i} | G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \omega_{-i}) dF(\omega_{-i} | \omega_i)$$

From the definition of the continuation equilibrium \tilde{m} , any array of incentive contracts and any effort in the support of $\pi_i(\cdot | G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \omega_i)$ must maximize agent i 's payoff conditional on $\{G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \omega_i\}$. In continuation equilibrium in the original game, the payoff for agent i is equal to

$$\text{Max}_{\alpha_i, e_i} \{y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot, \cdot | G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \cdot)) : \alpha_i^j \in G(\bar{\gamma}_i^j)(\mathcal{C}) \forall j \in \mathcal{J}, e_i \in E_i\}$$

For each $i \in \mathcal{I}$ and each $j \in \mathcal{J}$, the choice set of incentive contracts $\bar{\gamma}_i^j(\mathcal{A}_i^j)$ provided by the menu $\bar{\gamma}_i^j$ is equal to $G(\bar{\gamma}_i^j)(\mathcal{C})$ provided by the mechanism $G(\bar{\gamma}_i^j)$. It is then immediate that the continuation strategy \hat{q}_i is optimal for each $i \in \mathcal{I}$. Therefore, the array of continuation strategies $\hat{q} = (\hat{q}_1, \dots, \hat{q}_I)$ constitutes a continuation equilibrium relative to $\bar{\Gamma}$. Moreover, it shows

$$\begin{aligned} U(\hat{q}, \hat{\varepsilon}, \bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J, \omega_i) &= \\ \text{Max}_{\alpha_i, e_i} \{y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot, \cdot | G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \cdot)) : \alpha_i^j \in G(\bar{\gamma}_i^j)(\mathcal{C}_i^j) \forall j \in \mathcal{J}, e_i \in E_i\} &= \\ \int y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot, \cdot | \gamma, \cdot)) d\pi_i(\alpha_i, e_i | G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \omega_i) &= \\ U(\tilde{m}, \tilde{\varepsilon}, G(\bar{\gamma}_1^1), \dots, G(\bar{\gamma}_I^J), \omega_i) & \end{aligned}$$

If $(\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J)$ is equal to $(\hat{\gamma}_1^1, \dots, \hat{\gamma}_I^J)$,

$$U(\hat{q}, \hat{\varepsilon}, \hat{\gamma}_1^1, \dots, \hat{\gamma}_I^J, \omega_i) =$$

$$U(\tilde{m}, \tilde{\varepsilon}, G(\hat{\gamma}_1^1), \dots, G(\hat{\gamma}_I^J), \omega_i) = \\ U(\tilde{m}, \tilde{\varepsilon}, \tilde{\gamma}_1^1, \dots, \tilde{\gamma}_I^J, \omega_i)$$

Therefore, the equilibrium payoffs for agent i in the original game is reproduced when principals offer menus $(\hat{\gamma}_1^1, \dots, \hat{\gamma}_I^J)$.

Suppose that principal j offers $\hat{\gamma}^j$ given $\hat{\gamma}^{-j}$. The payoff for principal j is

$$V^j(\hat{\gamma}^j, \hat{\gamma}^{-j}, \hat{q}) = \\ \int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e | G(\hat{\gamma}_1^1), \dots, G(\hat{\gamma}_I^J), \omega) dF(\omega) = \\ \int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e | \tilde{\gamma}_1^1, \dots, \tilde{\gamma}_I^J, \omega) dF(\omega) = \\ V^j(\tilde{\gamma}^j, \tilde{\gamma}^{-j}, \tilde{m})$$

Therefore, the equilibrium payoff for principal j is preserved when principals offer menus $(\hat{\gamma}_1^1, \dots, \hat{\gamma}_I^J)$. Suppose that principal j deviates to some other array of menus, say $\gamma^j \in \bar{\Gamma}^j$.

$$V^j(\gamma^j, \hat{\gamma}^{-j}, \hat{q}) = \\ \int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e | G(\gamma^j), G(\hat{\gamma}^{-j}), \omega) dF(\omega) = \\ \int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e | \bar{\xi}^j(\gamma^j), \tilde{\gamma}^{-j}, \omega) dF(\omega) = \\ V^j(\bar{\xi}^j(\gamma^j), \tilde{\gamma}^{-j}, \tilde{m}, \tilde{\varepsilon}) \leq \\ V^j(\tilde{\gamma}^j, \tilde{\gamma}^{-j}, \tilde{m}, \tilde{\varepsilon})$$

Therefore, $(\hat{\gamma}_1^1, \dots, \hat{\gamma}_I^J) \in \bar{\Gamma}$ is the pure strategy equilibrium given the continuation equilibrium \hat{q} relative to $\bar{\Gamma}$ that preserves the equilibrium payoffs associated with a pure strategy equilibrium $(\tilde{\gamma}_1^1, \dots, \tilde{\gamma}_I^J) \in \Gamma$ given a continuation equilibrium \tilde{m} relative to Γ .

9.2 Proof of Theorem 2

We start with some basic definitions. Let us take two models for bilateral contracting Γ and $\bar{\Gamma}$ such that $\Gamma \succcurlyeq \bar{\Gamma}$. Let the associated continuation equilibria be \tilde{m} and \hat{q} respectively. $\pi(\cdot, \cdot | \gamma, \omega)_{\tilde{m}}$ and $\pi(\cdot, \cdot | \bar{\gamma}, \omega)_{\hat{q}}$ denotes the joint probability distributions on $\mathcal{A} \times E$ induced by

the two continuation equilibria when the collections of mechanisms are $\gamma = (\gamma_1^1, \dots, \gamma_I^J) \in \Gamma$ and $\bar{\gamma} = (\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J) \in \bar{\Gamma}$ respectively.⁹ \tilde{m} is said to *extend* \hat{q} if there is an embedding $\eta_i^j : \bar{\Gamma}_i^j \rightarrow \Gamma_i^j$ for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$ such that for all $\bar{\gamma} = (\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J) \in \bar{\Gamma}$

$$\pi(\cdot, \cdot | \bar{\gamma}, \omega)_{\hat{q}} = \pi(\cdot, \cdot | \eta(\bar{\gamma}), \omega)_{\tilde{m}}$$

where $\eta(\bar{\gamma}) = (\eta_1^1(\bar{\gamma}_1^1), \dots, \eta_I^J(\bar{\gamma}_I^J))$, $\pi(\cdot, \cdot | \bar{\gamma}, \omega)_{\hat{q}} = \pi_1(\cdot, \cdot | \bar{\gamma}, \omega)_{\hat{q}_1} \times \dots \times \pi_I(\cdot, \cdot | \bar{\gamma}, \omega)_{\hat{q}_I}$ and $\pi(\cdot, \cdot | \eta(\bar{\gamma}), \omega)_{\tilde{m}} = \pi_1(\cdot, \cdot | \eta(\bar{\gamma}), \omega)_{\tilde{m}_1} \times \dots \times \pi_I(\cdot, \cdot | \eta(\bar{\gamma}), \omega)_{\tilde{m}_I}$. It generalizes the idea behind direct mechanisms in the single principal problem such that principals explore more complex mechanisms in Γ that are not provided by the model of competition specified in bilateral contracting game relative to $\bar{\Gamma}$.

Proof. The method of the proof is to transform deviations that lie outside of the range of η into menus that they provide and then change the continuation equilibrium associated with those menus to coincide with the original equilibrium. The mapping ψ_i^j is used to associate each mechanism $\gamma_i^j \in \Gamma_i^j$ with the corresponding menu $\bar{\gamma}_i^j = \tau(\gamma_i^j(\mathcal{C}))$ for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$.

$\hat{q}_i(\cdot | \bar{\gamma}, \omega_i)$ maximizes agent i 's payoff given the other players' strategies \hat{q}_{-i} when the collection of menus is $\bar{\gamma} = (\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^J)$ and agent i 's valuation is ω_i . It follows that when the collection of mechanisms is $\gamma \in \Gamma$ and agent i 's valuation is ω_i , a continuation strategy that can induce probability distribution $\pi_i(\cdot, \cdot | \psi(\gamma), \omega_i)_{\hat{q}_i}$, maximizes agent i 's payoff. Choose a continuation equilibrium \tilde{m} relative to Γ satisfying, for all $i \in \mathcal{I}$, all $\gamma \in \Gamma$, and all $\omega_i \in \Omega_i$

$$\pi_i(\cdot, \cdot | \gamma, \omega_i)_{\tilde{m}_i} = \pi_i(\cdot, \cdot | \psi(\gamma), \omega_i)_{\hat{q}_i}$$

Let η^j be the mapping satisfying that $\eta^j(\bar{\gamma}_1^j, \dots, \bar{\gamma}_I^j) = (\eta_1^j(\bar{\gamma}_1^j), \dots, \eta_I^j(\bar{\gamma}_I^j))$ for all $(\bar{\gamma}_1^j, \dots, \bar{\gamma}_I^j) \in \bar{\Gamma}^j$. The strategy $\tilde{\sigma}^j$ chosen by principal j is induced by the mapping η^j given \hat{v}^j .

We begin with an equilibrium (\hat{v}, \hat{q}) relative to $\bar{\Gamma}$. The payoff for principal j who unilaterally deviates to some array of mechanisms $\gamma^j = (\gamma_1^j, \dots, \gamma_I^j)$ outside of $\eta^j(\bar{\Gamma}^j)$ is given by

$$\int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e | \gamma^j, \eta^{-j}(\bar{\gamma}^{-j}), \omega)_{\tilde{m}} \tilde{\sigma}^{-j}(\eta^{-j}(\bar{\gamma}^{-j})) dF(\omega)$$

⁹I make the notation explicitly contingent on those strategies to highlight the difference of two models of competition.

$$\begin{aligned}
&= \int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e | \psi^j(\gamma^j), \bar{\gamma}^{-j}, \omega)_{\hat{q}} \widehat{v}^{-j}(\bar{\gamma}^{-j}) dF(\omega) \\
&\leq V^j(\widehat{v}^j, \widehat{v}^{-j}, \widehat{\gamma}^{-j}, \widehat{q}) \\
&= V^j(\tilde{\sigma}^j, \tilde{\sigma}^{-j}, \tilde{m})
\end{aligned}$$

which proves that an equilibrium $(\widehat{v}, \widehat{q})$ relative to $\bar{\Gamma}$ is weakly robust.

9.3 Proof of Theorem 4

Proof. Fix an equilibrium $(\tilde{\sigma}, \tilde{m})$ relative to Γ . The equilibrium induces the probability distribution $\vartheta(\cdot, \cdot, \cdot | \gamma)$ on $Y \times E \times \Omega$ conditional on γ for all $\gamma \in \Gamma$. For all $\bar{\gamma} \in \bar{\Gamma}$ and all $s \in S = \Delta(Y \times E \times \Omega)$, $D(s : \bar{\gamma}) \subset \xi(\bar{\gamma})$ is defined as the subset of $\xi(\bar{\gamma})$ such that any collection of mechanisms γ in $D(s : \bar{\gamma})$ satisfies $\psi(\gamma) = \bar{\gamma}$ and generates the same equilibrium probability distribution $\vartheta(\cdot, \cdot, \cdot | \gamma) = s$ on $Y \times E \times \Omega$. Therefore, $\vartheta(\cdot, \cdot, \cdot | \gamma) = \vartheta(\cdot, \cdot, \cdot | D(s : \bar{\gamma}))$ for all $\gamma \in D(s : \bar{\gamma})$.

First, the mapping $\widehat{\zeta}$ is constructed. Let $\tilde{\sigma}(\cdot | B)$ be the probability distribution on Γ conditional on $B \subset \Gamma$ that is derived from $\tilde{\sigma}$ chosen by principals in the original game. If a collection of menus $\bar{\gamma} \in \bar{\Gamma}$ satisfies that $\xi^j(\bar{\gamma}^j) \in \text{supp } \tilde{\sigma}^j$ for all $j \in \mathcal{J}$, then $\widehat{\zeta}(\cdot | \bar{\gamma})$ satisfies, for all $s \in \Delta(Y \times E \times \Omega)$

$$\widehat{\zeta}(s | \bar{\gamma}) = \begin{cases} \tilde{\sigma}(D(s : \bar{\gamma}) | \xi(\bar{\gamma})) & \text{if } D(s : \bar{\gamma}) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

Suppose that a collection of menus $\bar{\gamma} \in \bar{\Gamma}$ is such that $\xi^j(\bar{\gamma}^j) \notin \text{supp } \tilde{\sigma}^j$ for some $j \in \mathcal{J}$ and $\xi^t(\bar{\gamma}^t) \in \text{supp } \tilde{\sigma}^t$ for all $t \in \mathcal{J} \setminus \{j\}$. Then choose some arbitrary array of mechanisms $\bar{\xi}^j(\bar{\gamma}^j)$ from $\xi^j(\bar{\gamma}^j)$. Let $D^{-j}(s : \bar{\gamma}^{-j})$ is the subset of $\xi^{-j}(\bar{\gamma}^{-j})$ satisfying that any collection of mechanisms $\gamma \in D(s : \bar{\gamma}) = \{\bar{\xi}^j(\bar{\gamma}^j)\} \times D^{-j}(s : \bar{\gamma}^{-j})$ generates the same equilibrium probability distribution $\vartheta(\cdot, \cdot, \cdot | \gamma) = s$ on $Y \times E \times \Omega$. In this case, the probability over $S = \Delta(Y \times E \times \Omega)$ conditional on $\bar{\gamma}$ satisfies, for all $s \in \Delta(Y \times E \times \Omega)$

$$\widehat{\zeta}(s | \bar{\gamma}) = \begin{cases} \tilde{\sigma}^{-j}(D^{-j}(s : \bar{\gamma}^{-j}) | \xi^{-j}(\bar{\gamma}^{-j})) & \text{if } D(s : \bar{\gamma}) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

$\tilde{\sigma}^{-j}(D^{-j}(s : \bar{\gamma}^{-j})|\xi^{-j}(\bar{\gamma}^{-j}))$ is derived from the equilibrium strategies $\tilde{\sigma}^{-j}$ used by all the principals except for principal j .

If a collection of menus $\bar{\gamma} \in \bar{\Gamma}$ does not belong to any of the two cases, then choose some arbitrary array of mechanisms $\bar{\xi}^j(\bar{\gamma}^j)$ from $\xi^j(\bar{\gamma}^j)$ all for $j \in \mathcal{J}$. One can pick up any degenerated probability distribution on S conditional on $\bar{\gamma}$ such that

$$\widehat{\zeta}(s|\bar{\gamma}) = 1 \quad (8)$$

where $\bar{\xi}(\bar{\gamma}) = (\bar{\xi}^1(\bar{\gamma}^1), \dots, \bar{\xi}^J(\bar{\gamma}^J))$ and $s = \vartheta(\cdot, \cdot, \cdot | \bar{\xi}(\bar{\gamma}))$.

Principal j 's strategy $\widehat{\nu}^j$ is induced by $\tilde{\sigma}^j$ through the mapping ψ^j . Agent i 's information partition $\widehat{\mathcal{P}}_i(\bar{\gamma}, \omega_i)$ is constructed given each $\bar{\gamma} \in \bar{\Gamma}$ and each $\omega_i \in \Omega_i$. $\cup P_n$ is S , where P_n s are the disjoint subsets of S . For any $s \in S$ such that $D(s : \bar{\gamma}) \neq \emptyset$, there exists a unique P_n satisfying $P_n = \{s\} \in \mathcal{P}_i(\bar{\gamma}, \omega_i)$. There exists the set $P_{n'}$ in $\widehat{\mathcal{P}}_i(\bar{\gamma}, \omega_i)$ that includes all the states satisfying $D(s : \bar{\gamma}) = \emptyset$. Agent i 's continuation strategy \widehat{g}_i is constructed as follows. For every $(\bar{\gamma}, s, \omega_i)$ in $\bar{\Gamma} \times S \times \Omega_i$, the continuation strategy \widehat{g}_i satisfies

$$\widehat{g}_i(\alpha_i, e|\bar{\gamma}, s, \omega_i) = \pi_i(\alpha_i, e_i|\gamma, \omega_i) \quad (9)$$

where γ is an arbitrary collection of mechanisms in $D(s : \bar{\gamma})$ if $D(s : \bar{\gamma}) \neq \emptyset$ otherwise γ is an arbitrary collection of mechanisms in $\xi(\bar{\gamma})$. The construction of \widehat{g}_i ensures the equilibrium condition: $\widehat{g}_i(\cdot|\bar{\gamma}, s, \omega_i) = \widehat{g}_i(\cdot|\bar{\gamma}, s', \omega_i)$ whenever $s \in P_i$ and $s' \in P_i$ for some $P_i \in \widehat{\mathcal{P}}_i(\bar{\gamma}, \omega_i)$ for each $i \in \mathcal{I}$ given any $\bar{\gamma} = (\bar{\gamma}_1^1, \dots, \bar{\gamma}_I^I) \in \bar{\Gamma}$ and any $\omega_i \in \Omega_i$. Each agent uses her continuation strategy constructed as above. For every $(\bar{\gamma}, s, \omega_i)$ in $\bar{\Gamma} \times S \times \Omega_i$, agent i 's payoff is

$$\begin{aligned} U(\widehat{g}, \bar{\gamma}, s, \omega_i) &= \\ &\int u(a(e), e, \omega_i) d\alpha(a) d\widehat{g}(\alpha, e|\bar{\gamma}, s, \omega) dF(\omega_{-i}|\omega_i) = \\ &\int u(a(e), e, \omega_i) d\alpha(a) \pi(\alpha, e|\gamma, \omega_i) dF(\omega_{-i}|\omega_i) = \\ &U(\tilde{m}, \gamma, \omega_i) \end{aligned}$$

The profile of continuation strategies $\widehat{g} = (\widehat{g}_1, \dots, \widehat{g}_I)$ is a continuation equilibrium by the same logic used in proving theorem 1. The construction of the mapping $\widehat{\zeta}$ and the

continuation equilibrium \widehat{g} ensure that for any ω_i and any collection of mechanisms γ offered with positive probability, there exists a pair of a collection of menus and a state such that agent i 's equilibrium payoff in the new game is equal to the one in the original game when the valuation and the array of mechanisms are ω_i and γ . This pair of the collection of menus and the state occurs with positive probability by the construction of the mapping $\widehat{\zeta}$ and principals' strategies, $\widehat{v} = (\widehat{v}^1, \dots, \widehat{v}^J)$.

Suppose that principal j uses \widehat{v}^j given \widehat{v}^{-j} . Principal j 's expected utility is

$$\begin{aligned}
V^j(\widehat{v}^j, \widehat{v}^{-j}, \widehat{g}) &= \tag{10} \\
&\int v^j(a(e), e, \omega) d\alpha(a) \widehat{g}(\alpha, e | \overline{\gamma}, s, \omega) d\widehat{\zeta}(s | \overline{\gamma}) d\widehat{v}(\overline{\gamma}) dF(\omega) = \\
&\int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e | \gamma, \omega_i) d\tilde{\sigma}(D(s : \overline{\gamma}) | \xi(\overline{\gamma})) d\tilde{\sigma}(\xi(\overline{\gamma})) dF(\omega) = \\
&\int v^j(y, e, \omega) d\vartheta(y, e, \omega | D(s : \overline{\gamma})) d\tilde{\sigma}(D(s : \overline{\gamma}) | \xi(\overline{\gamma})) d\tilde{\sigma}(\xi(\overline{\gamma})) = \\
&V^j(\tilde{\sigma}^j, \tilde{\sigma}^{-j}, \tilde{m})
\end{aligned}$$

γ in the third line in (10) is an arbitrary collection of mechanisms in $D(s : \overline{\gamma})$. The second equality holds because of the definition of $(\widehat{\zeta}, \widehat{v}, \widehat{g})$. The third equality holds because every collection of mechanisms in $D(s : \overline{\gamma})$ generates the same equilibrium probability distribution on $Y \times E \times \Omega$. The last equality holds because of the definition of $V^j(\tilde{\sigma}^j, \tilde{\sigma}^{-j}, \tilde{m})$. Moreover, it is immediate from the construction of $(\widehat{\zeta}, \widehat{v}, \widehat{g})$ that any array of menus in $\text{supp } \widehat{v}^j$ generates the same payoff.

Finally, we need to prove that \widehat{v}^j is an equilibrium strategy for principal j given \widehat{v}^{-j} . Suppose that principal j deviates to some $\overline{\gamma}^j \notin \text{supp } \widehat{v}^j$. Let (γ^j, γ^{-j}) be a collection of mechanisms in Γ such that $\gamma^j = \overline{\xi}^j(\overline{\gamma}^j)$ and $(\gamma^j, \gamma^{-j}) \in D(s : \overline{\gamma}^j, \overline{\gamma}^{-j}) = \{\overline{\xi}^j(\overline{\gamma}^j)\} \times D^{-j}(s : \overline{\gamma}^{-j})$ with $\overline{\gamma}^{-j} = \psi^{-j}(\gamma^j)$. Let $\tilde{\sigma}^{-j}(D^{-j}(s : \overline{\gamma}^{-j}))$ be $\tilde{\sigma}^{-j}(D^{-j}(s : \overline{\gamma}^{-j}) | \xi^{-j}(\overline{\gamma}^{-j})) \times \tilde{\sigma}^{-j}(\xi^{-j}(\overline{\gamma}^{-j}))$. The payoff for principal j is

$$\begin{aligned}
V^j(\overline{\gamma}^j, \widehat{v}^{-j}, \widehat{g}) &= \tag{11} \\
&\int v^j(a(e), e, \omega) d\alpha(a) \widehat{g}(\alpha, e | \overline{\gamma}^j, \overline{\gamma}^{-j}, s, \omega) d\widehat{\zeta}(s | \overline{\gamma}^j, \overline{\gamma}^{-j}) d\widehat{v}^{-j}(\overline{\gamma}^{-j}) dF(\omega) =
\end{aligned}$$

$$\int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e | \gamma^j, \gamma^{-j}, \omega) d\tilde{\sigma}^{-j}(D^{-j}(s : \bar{\gamma}^{-j})) dF(\omega) =$$

$$\int v^j(y, e, \omega) d\vartheta(y, e, \omega | D(s : \bar{\gamma}^j, \bar{\gamma}^{-j})) d\tilde{\sigma}^{-j}(D^{-j}(s : \bar{\gamma}^{-j})) =$$

$$V^j(\gamma^j, \tilde{\sigma}^{-j}, \tilde{m})$$

Since $\bar{\gamma}^j$ is not in $\text{supp } \hat{\nu}^j$, γ^j is not in $\text{supp } \tilde{\sigma}^j$. The definition of $(\hat{\zeta}, \hat{\nu}, \hat{g})$ makes the second equality hold. The third equality holds because any collection of mechanisms in $D(s : \bar{\gamma}^j, \bar{\gamma}^{-j})$ generates the same equilibrium probability on $Y \times E \times \Omega$. Consider the last equality. By (7), $D^{-j}(s : \bar{\gamma}^{-j})$ is the subset of $\xi^{-j}(\bar{\gamma}^{-j})$ satisfying that any collection of mechanisms $\gamma \in D(s : \bar{\gamma}^j, \bar{\gamma}^{-j}) = \{\bar{\xi}^j(\bar{\gamma}^j)\} \times D^{-j}(s : \bar{\gamma}^{-j})$ generates the same equilibrium probability distribution $\vartheta(\cdot, \cdot, \cdot | \gamma) = s$ on $Y \times E \times \Omega$. So, $\vartheta(\cdot, \cdot, \cdot | \gamma) = \vartheta(\cdot, \cdot, \cdot | D(s : \bar{\gamma}^j, \bar{\gamma}^{-j})) = s$ for all $\gamma \in D(s : \bar{\gamma}^j, \bar{\gamma}^{-j})$. The last equality therefore holds immediately because of the definition of $V^j(\gamma^j, \tilde{\sigma}^{-j}, \tilde{m})$. Since γ^j is not in $\text{supp } \tilde{\sigma}^j$, $V^j(\bar{\gamma}^j, \hat{\nu}^{-j}, \hat{g}) = V^j(\gamma^j, \tilde{\sigma}^{-j}, \tilde{m}) \leq V^j(\tilde{\sigma}^j, \tilde{\sigma}^{-j}, \tilde{m}) = V^j(\hat{\nu}^j, \hat{\nu}^{-j}, \hat{g})$. Therefore, there are no arrays of menus outside of $\text{supp } \hat{\nu}^j$ that generate strictly higher payoff for principal j . Therefore, $\hat{\nu}^j$ is a best response for principal j given $\hat{\nu}^{-j}$.

References

- [1] Bernheim B. D. and M. D. Whinston (1986), "Menu Auctions, Resource Allocation, and Economic Influence," *Quarterly Journal of Economics*, 101, 1-31.
- [2] Dixit, A., G. Grossman, and E. Helpman (1997), "Common Agency and Coordination: General Theory and Application to Government Policy Making," *Journal of Political Economy*, 105, 752-769.
- [3] Epstein, L. and M. Peters (1999), "A Revelation Principle for Competing Mechanisms," *Journal of Economic Theory*, 88, 119-160.
- [4] Klemperer, P. D. (2002), "What Really Matters in Auction Design," *Journal of Economic Perspectives*, 16, 169-89.

- [5] Klemperer, P. D. and M. A. Meyer (1989), "Supply Function Equilibria in Oligopoly under Uncertainty," *Econometrica*, 57, 1243-1277.
- [6] Martimort, D. and L. Stole (2002), "The Revelation and Delegation Principles in Common Agency Games," *Econometrica*, 70, 1659-73.
- [7] McAfee, P. (1993), "Mechanism Design by Competing Sellers," *Econometrica*, 61, 1283-1312.
- [8] Peters, M. (2001), "Common Agency and the Revelation Principle," *Econometrica*, 69, 1349-1372.
- [9] Prat, A. and A. Rustichini (2002), "Games Played through Agents," forthcoming in *Econometrica*
- [10] Segal, I. (1999), "contracting with Externalities," *Quarterly Journal of Economics*, 114, 337-388.
- [11] Segal, I. and M. Whinston (2003), "Robust Predictions for Bilateral contracting with Externalities," *Econometrica*, 71, 757-791.