Identifying Collusion in English Auctions

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Abstract

We develop a fully nonparametric identification framework and a test of collusion in ascending bid auctions. Assuming efficient collusion, we show that the underlying distributions of values can be identified despite collusive behaviour when there is at least one known competitive bidder. We propose a nonparametric estimation procedure for the distributions of values and a bootstrap test of the null hypothesis of competitive behaviour against the alternative of collusion. Our framework allows for asymmetric bidders, and the test can be performed on individual bidders.

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1 Introduction

Collusion in auctions is an antitrust violation, but is nevertheless a pervasive phenomenon. It has been subject to many empirical studies. However, much of the research has focused on the sealed-bid, first-price auction format. For example, Porter and Zona

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(1993) and Bajari and Ye (2003) have studied collusion in highway procurement, while Porter and Zona (1999) and Pesendorfer (2000) have studied collusion in school milk procurement.[1]

There has been relatively less empirical or econometric work on collusion in *open* (or *English*) auctions, partly because of the dominance of the sealed-bid format in public procurement and sales.[2] The arrival of the Internet has greatly reduced the costs of bringing buyers and sellers together, and thus contributed to the increase in popularity of open auctions.

In this paper, we provide a structural nonparametric identification, estimation and testing framework for collusion in open auctions. The analysis focuses on the commonly accepted theoretic model of such auctions, namely the button (or thermometer) model, where the price is risen continuously and bidders drop out irrevocably. This model is becoming increasingly relevant for the auctions conducted over the Internet. The reason for this is the availability (and popularity) of electronic bidding agents that update bids continuously on bidders' behalf, which effectively implements the button model.

We make the most often exploited assumption: bidders draw their values independently (the IPV framework), however, allowing for *bidder asymmetries*. As the benchmark, and also the first step in our approach, we consider a model where there is no collusion. It is assumed that all the loosing bids are observable. The main difficulty with identification and estimation of value distributions is the *censoring problem*: while the losing bids reveal bidder values, the winning value is censored. Our approach to de-censoring is based on the Nelson-Aalen estimator originally developed in the competing risks literature. We derive a simple formula that allows one to identify the value distribution of a particular bidder using only its losing bids and the losing bids of its highest rival.

Our main contribution is to extend this de-censoring technique to potentially colluding bidders. We restrict attention to collusion through cover (or phantom) bidding, a commonly used form of collusion in auctions.[3] In open auctions, the gains from collusion are maximal when the cartel members do not bid higher than the highest dropout price of their competitive rivals. In reality, they still may bid higher in order to conceal collu-

[1] See a survey by Harrington (2008) for more examples.

[2] One exception is Baldwin et al. (1997), who have studied collusion is timber auctions.

[3] Collusion in auctions can take other forms, notably a market division agreement. See Hendricks and Porter (1989). Pesendorfer (2000) presents evidence that collusion takes different forms in highway procurement auctions in Florida and Texas.

sion. The exact nature of cover bidding is not needed for our analysis as only the leading cartel bid is used. For example, we allow non-participation, where instead of submitting a low bid, the cover bidder does not bid at all.[4,]

Our result relies on several identifying assumptions. First, we assume that values are drawn independently, however, allowing for nonidentical distributions. The latter is important because the cartel is usually stronger on average than any of the non-cartel bidders.

Second, it is assumed that only one serious bid is submitted by the cartel, by a bidder that we call the *cartel leader*. The cartel leader is assumed to be selected *efficiently*, i.e. as the bidder with the highest valuation. This efficiency assumption is commonly made in the empirical literature on auctions, and is also supported by auction theory, as we explain in the next section.

Third, it is assumed that there is at least one competitive firm bidding against the cartel. This is often the case empirically, as e.g. in Porter and Zona (1993), Porter and Zona (1999), and Baldwin et al. (1997). Apart from this, the composition of the cartel does not need to be known. It is only important that the cartel leader bids competitively against the non-cartel firms.[5]

The cartel leader's value is censored from above by the competitive bid. At the same time, being the maximal value among the cartel bidders, it is censored from *below* by the second-highest cartel value. So unlike the competitive setup, here we have a joint censoring of the value both from above and below. Nevertheless, we show that the value distribution can be de-censored for each bidder in the cartel. This is because, as we show, the selection mechanism is identifiable under efficient collusion. This identification result is constrictive in that it gives a closed-form formula for the de-censored distribution of the values of the cartel members that is simple to estimate nonparametrically.

In our analysis, the cartel set should be understood as a *suspect* set. If competitive firms are mistakenly included in the cartel, the identification of the values of the colluders is unaffected as long as there is at least one competitive firm outside the cartel. Empirical studies often provide direct evidence as to who might be a potential colluder. This evidence often allows to plausibly argue that certain firms are "clean", i.e. did not participate in the conspiracy. Sometimes the cartel composition is known, as the defendant in an antitrust case as in Porter and Zona (1993) and Porter and Zona (1999). However,

[4]See, e.g. Porter and Zona (1993) and Baldwin et al. (1997).

[5]It is also permissible that the fringe firms collude among themselves.

the strength of our approach is that it works under minimal knowledge concerning the composition of the cartel.

As we have argued, regardless of whether a bidder is competitive or not, its value distribution is identifiable through our de-censoring approach. This allows us to construct the counterfactual distribution of its bids under competition, even if the bidder's actual behaviour is collusive. If the bidder is competitive, then the counterfactual and actual distributions will coincide. However, if the bidder is collusive, we show that the counterfactual competitive bid distribution stochastically dominates the actual collusive one. This allows us to design a formal statistical test of the null hypothesis of competitive bidding against the alternative of collusive bidding. The test can be applied individually bidder by bidder, or can be applied jointly to a group of bidders.

Our test is initially developed at the individual bidder level. However, in combination with Bonferroni-type sequential hypothesis testing such as Holm (1979), it leads to a simple estimator of the composition of the cartel.[6] In our setting, the Holm-Bonferroni procedure works as follows. First, each bidder in the suspect set is tested and the p-value of the test recorded. Second, the p-values are ordered from smallest to highest. The bidders are then tested sequentially at appropriately adjusted levels of significance. If the competitive behaviour of the suspect bidder with the smallest p-value is not rejected, then the procedure terminates with no collusion found. If not, then this bidder is classified as a colluder, and the procedure moves to the next bidder in the order. This bidder is tested at a higher level of significance, and is included in the cartel following rejection. If no rejection occurs, then the test finds no presence of a cartel, as it is impossible to have a single-firm cartel. Continuing in this fashion until termination, the procedure results in an estimated cartel set with at least two bidders. The probability of one or more false bidder inclusions in the cartel is controlled overall at a predetermined level of significance, e.g. 5%. Moreover, the estimator of the cartel set is consistent.[7]

Once the collusive set has been estimated, we can proceed to estimate the collusive damages. For each colluding bidder, we can estimate its value distribution, which determines its dropout prices under competition. This allows us to recover the distribution of the auction price if all bidders were competitive, and to compare this counterfactual distribution with the actual distribution of the prices. For example, one could estimate

[6] This approach is also adopted in Schurter (2017) to estimate the composition of the cartel in a first-price auction.

[7] Recently, Coey et al. (2014) considered placing bounds on collusive damages and proposed an approach based on bidder exclusion.

the average loss of revenue due to collusion, and other statistics of the loss' distribution.

We are not aware of any previous research on nonparametric identification of collusion in open auctions. We believe our paper is the first one to investigate this issue. Our parallel contribution is that we propose full identification of model primitives under collusion. This can be used to address other important policy questions such as, for example, the optimal reserve price under collusion.

Relation to the existing literature

A common approach in the empirical literature on collusion in auctions is to use different bid responses to exogenous variation under collusion and competition. Porter and Zona (1993) study collusion in first-price highway procurement auctions conducted by the New York State Department of Transportation. They use measures of capacity and utilization rates as explanatory variables, and develop a likelihood-based model stability test across low and high bid ranks. The cartel composition is known in their case as they have access to court records. They find that parameter estimates are stable for the competitive group, but not for the cartel, which provides strong reduced-form evidence for collusion in the form of phantom bidding.

In another influential paper, Porter and Zona (1999) consider collusion in Ohio school milk auctions. They find that while the probability of submitting a bid falls with distance for non-defendant diaries, it increases for the defendants. Also, bid levels increase with distance for the non-defendants, but decrease for the defendants. These reduced-form finding convincingly point to collusion among the defendants, in the form territorial allocation.

Bajari and Ye (2003) adopt a structural approach in their study of collusion in highway procurement. The essence of their approach is to derive high-level testable predictions of the competitive model such as conditional independence and exchangeability, and build a statistical test based on these predictions. The main structural assumption is that the cartel is efficient, as in our paper. An extension of Bajari and Ye's approach to English auctions is difficult because censoring of the highest valuation implies that the dropout prices are correlated even under competition.

Aryal and Gabrielli (2013) consider a different test of collusion in first-price auctions. They exploit the variation in the number of bidders to argue that only the true model (competition or collusion) results in an invariant distributions of bidder values. Also for first-price auctions, Schurter (2017) exploits the potential presence of an exogenous shifter in the level of competition, and develops a test of collusion in first-price auctions based on the independence between the valuations and the shifter if the bidder is competitive.

There is very little work on collusion in open English auctions. Baldwin et al. (1997) considered collusion in US Forest Service Timber auctions. They consider a symmetric setting where bidders draw values from the same parametric distribution, and assume that the cartel is efficient. Within their parametric specifications, they compare likelihoods of competitive and collusive models and find support for collusion.

Asker (2010) estimates damages from collusion in a structural model of a knockout auction of stamp dealer cartel.

Athey and Haile (2002) is a fundamental paper on identification in auctions, and provides a proper perspective on our identification results. Without collusion, and in the IPV framework as in our paper, it is known that the asymmetric ascending bid auction is identifiable even if only the winning bids are observable. This has been established in Athey and Haile (2002), building on the results for competing risks in Meilijson (1981). This approach has been recently extended by Komarova (2013). However, feasible nonparametric estimators have not been developed due to the complex nature of the identification arguments.[8] It is not known if the model is identifiable from the winning bids in the presence of collusion.

Our estimator *in the absence of collusion* is based on a well-known Nelson-Aalen estimator for models with random censoring.[9] However, its application to auctions is novel as is our approach to the identification and estimation of the value distributions under collusion.

We adopt the button model of the English auction. Haile and Tamer (2003) emphasize that losing bids do not necessarily reflect true values because of jump bidding in many real-world open auctions. Be this as it may in the traditional open auctions, the arrival of the Internet has opened door to new ascending-bid auctions that, as we have argued, conform more closely to the original "button" model considered in the theoretical

[8]The identification using winning bids only relies on Pfaffian integral equations, which are very difficult to solve even numerically. See Brendstrup and Paarsch (2007), who instead appeal to parametric flexible-form maximum likelihood estimation. We should also mention that outside the IPV framework, the model is not identifiable even under symmetry. A recent paper by Aradillas-López et al. (2011) addresses partial identification of this model.

[9]See e.g. the discussion in Section 20.15 in van der Vaart (1998).

literature.

Our main structural identification assumption is that the cartel is efficient. This assumption is commonly used in the empirical literature on auctions, e.g. Bajari and Ye (2003), Baldwin et al. (1997). Auction theory supports it as well: Graham and Marshall (1987) show that, if the bidding cartel is able to distribute the spoils of collusion ex ante, it can efficiently select the cartel leader using an open knockout auction. In addition, Mailath and Zemsky (1991) show that efficient collusion can be sustained through appropriate ex-post side payments between the cartel members if the values are independent, while Hendricks et al. (2008) show that this continues to be true if values are affiliated.[10] When cartel bidders are symmetric, a simple knockout auction exists that selects the leader efficiently and balances the budget ex post.

2 Identification under competition

In the baseline competitive model, we consider a standard independent private values (IPV) setting where there are N bidders participating at an auction. The set of bidders is denoted as $\mathcal{N} = \{1, ..., N\}$.

Assumption 1 (IPV)**.** *Each bidder $i \in \mathcal{N}$ draws its value independently from a cumulative distribution $F_i(\cdot)$ supported on $[0, \overline{v}]$.*

We allow the distributions F_i to be different across bidders, but assume that the support $[0, \overline{v}]$ is the same for all bidders. The density of F_i is denoted as f_i.

In an ascending button auction, only the dropout prices of the losing bidders are equal to valuations in a dominant strategy equilibrium. The valuation of the winner is censored from below by the highest dropout price among the losing bidders. For any bidder i, let V_i denote its value, and let V_{-i} denote the maximum value of its rivals, $V_{-i} = \max_{j \neq i} V_j$. The distribution of V_{-i} is denoted as $F_{-i}(\cdot)$. The indicator variable $w_i \in \{0, 1\}$ is equal to 1 if bidder i wins the auction, and is equal to 0 if he looses. If $w_i = 0$, V_i is observable, while V_i is censored from above by V_{-i} when $w_i = 1$. Let $g_i(v|w_i = 0)$ be the density of i's bids, or equivalently, the values conditional on *losing* the auction. It is directly identifiable from the data.

[10] However not if values are common. See Hendricks et al. (2008).

We now show how to recover F_i. Since V_i and V_{-i} are assumed to be independent, the Bayes rule yields

$$g_i(v|w_i = 0) = \frac{f_i(v)(1 - F_{-i}(v))}{\mathbb{P}(w_i = 0)}$$
$$\implies f_i(v) = \frac{g_i(v|w_i = 0)\mathbb{P}(w_i = 0)}{1 - F_{-i}(v)}.$$

Dividing both sides of the last equation by $1 - F_i(v)$, we obtain

$$\frac{f_i(v)}{1 - F_i(v)} = \frac{g_i(v|w_i = 0)\mathbb{P}(w_i = 0)}{(1 - F_i(v))(1 - F_{-i}(v))}. \tag{1}$$

Our *key insight* is that the function that appears on the right-hand side in the denominator of (1) is directly identifiable. The independence between V_i and V_{-i} implies that

$$(1 - F_i(v))(1 - F_{-i}(v)) = \mathbb{P}(\min\{V_i, V_{-i}\} \geq v).$$

However,

$$B_i = \min\{V_i, V_{-i}\} = w_i V_{-i} + (1 - w_i)V_i$$

is in fact equal to bidder i's actual bid (whether losing or winning), and is directly observable. Its distribution,

$$G_i(v) \equiv \mathbb{P}(B_i \leq v),$$

is therefore directly identifiable from the data. Thus, the result in equation (1) can be equivalently stated as

$$\frac{f_i(v)}{1 - F_i(v)} = \frac{g_i(v|w_i = 0)\mathbb{P}(w_i = 0)}{1 - G_i(v)}, \tag{2}$$

where the expression on the right-hand side involves only terms that can be directly estimated from the data.

It will prove convenient to define

$$G_i^0(b) \equiv P(B_i \leq b, w_i = 0) = G_i(b|w_i = 0)\mathbb{P}(w_i = 0),$$

and its derivative

$$g_i^0(b) \equiv \frac{dG_i^0(b)}{db}.$$

We can now re-state the identification result in (2) as

$$-\frac{d\log(1-F_i(v))}{dv} = \frac{g_i^0(v)}{1-G_i(v)}.$$

The left-hand side of this equation can be recognized as a *full derivative*, so we can integrate this equation and recover the distribution of i's values $F_i(\cdot)$. The result is given in the proposition below.

Proposition 1 (Identification under competition)**.** *Under Assumption 1, we have*

$$F_i(v) = 1 - \exp\left(-\int_0^v \frac{dG_i^0(u)}{1-G_i(u)}\right). \tag{3}$$

This result can be viewed as an adaptation of the well-known Nelson-Aalen estimator originally developed for cumulative hazard functions (Nelson, 1969, 1972; Aalen, 1978) to ascending auctions. The functional that appears on the right-hand side of (3) will be used repeatedly in the sequel. It is defined, for any two functions $H_1(\cdot)$ and $H_2(\cdot)$, as[11]

$$\psi(H_1, H_2)(v) \equiv 1 - \exp\left(-\int_0^v \frac{dH_1(u)}{1-H_2(u)}\right). \tag{4}$$

Note that using the definition in (4), the result in (3) can be stated as $F_i(v) = \psi(G_i^0, G_i)(v)$.

3 Collusion

In this section, we show that the distributions of bidder valuations are identifiable even in the presence of collusion. We assume that a subset of bidders potentially forms a *bidding cartel.* The identification is shown under a number of assumptions.

First, we assume that the cartel is not all inclusive. That is, it is known to the researcher that at least one bidder behaves competitively, i.e. bids up to its true value.[12] Denote the set of known competitive bidders as $\mathcal{N}_{com}$.

Assumption 2 (Competitive bidder)**.** *There is at least one known competitive bidder, i.e. the set $\mathcal{N}_{com}$ is non-empty.*

[11] This functional is well-defined when H_1 has bounded variation.
[12] This assumption can be relaxed, as we remark in the sequel.

We assume that some bidders *may be* colluding. The colluding bidders are necessarily contained in

$$\mathcal{N}_{col} = \mathcal{N} \backslash \mathcal{N}_{com}.$$

We shall sometimes refer to $\mathcal{N}_{col}$ as the *suspect set*, as this set may also include some firms that are in fact competitive. It is important to note that the set of actually colluding bidders $\mathcal{C} \subseteq \mathcal{N}_{col}$ is not a priory known. We also allow for no collusion at all, in which case $\mathcal{C} = \emptyset$. Our identification approach is based on the idea that a cartel firm still behaves competitively if it is the *cartel leader*, i.e. the designated highest bidder from the cartel.

Second, we restrict attention to *efficient collusion*, where the ring (cartel) leader is the bidder with the highest valuation of the item.[13]

Assumption 3 (Efficient collusion)**.** *Cartel leader's valuation is equal to* $\max_{k \in \mathcal{C}} V_k$.

Let $\ell_i = 1$ indicate the event that bidder i has the leading (maximum) value in the suspect set $\mathcal{N}_{col}$, otherwise $\ell_i = 0$. This obviously includes the event when bidder i is the cartel leader under efficient collusion, however also requires i's value to be higher than any of the competitive bidders' values in $\mathcal{N}_{col}$. Note that together with our assumption that the distributions $F_i(\cdot)$ have the same support, efficient collusion implies that each suspect member has a positive probability of being the leader, i.e. $\mathbb{P}(\ell_i = 1) > 0$. By the Bayes rule,

$$\begin{aligned} f_i(v|\ell_i = 1) &= \frac{\mathbb{P}(\ell_i = 1 | V_i = v) f_i(v)}{\mathbb{P}(\ell_i = 1)} \\ \implies f_i(v) &= \frac{\mathbb{P}(\ell_i = 1) f_i(v|\ell_i = 1)}{\mathbb{P}(\ell_i = 1 | V_i = v)}. \end{aligned} \tag{5}$$

Conditional on being a leader, i bids competitively against the competitive fringe $\mathcal{N}_{com}$. This implies that the density $f_i(v|\ell_i = 1)$ is identifiable using the results in the previous section, i.e. by considering i's bids that are both leading ($\ell_i = 1$) and loosing in the action ($w_i = 0$) against the competitive fringe. Let

$$V_{com} \equiv \max_{k \in \mathcal{N}_{com}} V_k$$

[13] This assumption is plausible in empirical applications and frequently made in the literature. See e.g. Bajari and Ye (2003).

be the maximum value in the competitive fringe $\mathcal{N}_{com}$. In parallel to (3) in the previous section, the distribution of i's values conditional on leading the cartel,

$$F_i^{\ell}(v) \equiv F_i(v|\ell_i = 1),$$

is identifiable through the de-censoring formula

$$F_i^{\ell}(v) = \psi\big(G_i^{0,\ell}, G_i^{\ell}\big)(v), \tag{6}$$

where the distributions $G_i^{0,\ell}(b)$ and $G_i^{\ell}(b)$ are now conditional on being the cartel leader,

$$G_i^{0,\ell}(b) = \mathbb{P}(b_i \leq b, w_i = 0|\ell_i = 1), \qquad G_i^{\ell}(b) = \mathbb{P}(b_i \leq b|\ell_i = 1).$$

Note that both $G_i^{0,\ell}(b)$ and $G_i^{\ell}(b)$ are identifiable from the data.

Continuing the identification argument, the selection probability $\mathbb{P}(\ell_i = 1|V_i = v)$ that appears in (5) is not directly identifiable. In order to apply the above result, we propose a transformation that does not involve $\mathbb{P}(\ell_i = 1|V_i = v)$. Dividing both sides of (5) by $F_i(v)$, we obtain

$$\frac{F_i'(v)}{F_i(v)} = \frac{\mathbb{P}(\ell_i = 1) f_i(v|\ell_i = 1)}{\mathbb{P}(\ell_i = 1|V_i = v) F_i(v)}. \tag{7}$$

Under independence and efficient collusion, the leader selection probability is simply the product of the CDFs of bidders in $\mathcal{N}_{col} \backslash \{i\}$,

$$\mathbb{P}(\ell_i = 1|V_i = v) = \prod_{j \in \mathcal{N}_{col} \backslash \{i\}} F_j(v) \tag{8}$$

$$\implies P(\ell_i = 1|V_i = v) F_i(v) = \prod_{j \in \mathcal{N}_{col}} F_j(v) \equiv F_{col}(v) \tag{9}$$

where $F_{col}(v)$ is the distribution of the maximum value V_{col} in the suspect set,

$$V_{col} \equiv \max_{k \in \mathcal{N}_{col}} V_k.$$

Since the bidder with valuation V_{col} bids competitively against the maximum value V_{com} in competitive fringe $\mathcal{N}_{com}$, the distribution $F_{col}(v)$ is identifiable by de-censoring in parallel to (3) from the previous section:

$$F_{col}(v) = \psi(G^0_{col}, G_{col})(v), \tag{10}$$

where

$$G^0_{col}(u) = \mathbb{P}\{\min\{V_{com}, V_{col}\} \le u; w_{col} = 0\}, \qquad G_{col}(u) = \mathbb{P}\{\min\{V_{com}, V_{col}\} \le u\}.$$

Here $w_{col} \in \{0, 1\}$ indicates whether or not the suspect leader wins the auction. Note that both G^0_{col} and G_{col} are identifiable because $\min\{V_{com}, V_{col}\}$ is observable.

Substituting (9) into (7), we obtain a differential equation for $F_i(v)$ that only involves identifiable objects,

$$\frac{dF_i(v)}{F_i(v)} = \frac{dF^\ell_i(v)}{F_{col}(v)}. \tag{11}$$

This differential equation can be integrated backwards using the boundary condition $F_i(\overline{v}) = 1$ to yield a unique solution given in the proposition below, which is our main result in this section.

Proposition 2 (Identification under efficient collusion)**.** *Under Assumptions 1–3, the distributions $F_i(\cdot)$ are identifiable. The identification of $F_i(\cdot)$ for the known competitive bidders is unaffected and proceeds according to* (3)*, as before. The identification of $\{F_i(\cdot) : i \in \mathcal{N}_{col}\}$ can be performed according to*

$$F_i(v) = \exp\left(-\int_v^\infty \frac{dF^\ell_i(u)}{F_{col}(u)}\right), \tag{12}$$

where the distributions $F^\ell_i(v)$ and $F_{col}(v)$ are identifiable from the previous step according to (6) *and* (10) *respectively.*

The intuition behind this identification result can be summarized as follows. First, even though bidders in the cartel may submit noncompetitive "cover" bids, the cartel leader bids competitively against any competitive bidder (i.e. any bidder in the set $\mathcal{N}_{com}$). In particular, we use the fact that it bids competitively against the highest bidder in $\mathcal{N}_{com}$. The implication of this observation is that, *conditionally on being a cartel leader*, the bidder's behavior in the auction is in fact competitive. The de-censoring approach can be used to identify, for any suspect bidder, the distribution of valuations conditionally on leading the cartel.

Second, under our assumption that the cartel is efficient, the valuation of the cartel leader is censored from below. We have shown that the de-censoring approach can be suitably extended to uncover the marginal distribution of bidder values even in this case.

Assumption 2, which requires that there is at least one known competitive bidder, can be relaxed. If the seller is an active participant in the auction, then the seller's bid can be used instead of the maximum competitive bid for the purposes of identification, as long as it is independent of the maximum cartel value. The seller may or may not know that it is facing a cartel, and may or may not bid optimally. It would only be required that the seller's bids have support $[\underline{b}, \infty)$ for some $\underline{b} \geq 0$.

3.1 Identifying collusion

The result in Proposition 2 can be used as a basis for a test of collusion. Regardless of whether bidder $i \in \mathcal{N}_{col}$ is colluding or not, and regardless of the potential presence of an unknown (but efficient) cartel, we can identify the *predicted* distribution of bidder i's bids if i were competitive. It is assumed that, if there is a cartel, it continues to operate with bidder i excluded. This (potentially counterfactual) distribution is denoted as $G_i^{pred}(v)$. As V_i, V_{-i} are independent if bidder i is competitive, the upper CDF of i's bid $B_i = \min\{V_i, V_{-i}\}$ is given by the product

$$\begin{aligned} 1 - G_i^{pred}(v) &= (1 - F_i(v))(1 - F_{-i}(v)), \\ \implies G_i^{pred}(v) &= 1 - (1 - F_i(v))(1 - F_{-i}(v)). \end{aligned} \tag{13}$$

In this formula, $F_i(v)$ is identifiable according to (12), and $F_{-i}(v)$, the distribution of the maximum of all bidder values excluding bidder i, is identifiable by an analogue to (3):

$$F_{-i}(v) = \psi(G_i(\cdot|w_i = 1)\mathbb{P}(w_i = 1), G_i(\cdot))(v). \tag{14}$$

Alternatively, since all the individual CDFs have been identified, one can take

$$F_{-i}(v) = \prod_{j \neq i} F_j(v). \tag{15}$$

It will be more convenient to use the latter expression for F_{-i}.

The actual behavior of bidder $i \in \mathcal{N}_{col}$ may be collusive. We now detail the assumptions on the bidding strategy of a *cover bidder*, i.e. a cartel member who is not the cartel leader. Let $h = ((i_1, p_1), ..., (i_k, p_k))$ be a dropout history, where $i_1, ..., i_k \in \mathcal{N}$ indicate the identities of those k bidders that have dropped out, and $p_1, ..., p_k$ denote their respective dropout prices. If there are no dropouts yet, we let $h = \emptyset$. A bidding strategy $B_i^*(v_{\mathcal{C}}, h)$ of a cover bidder specifies the maximum price up to which the bidder is willing to stay in the auction, given the history h, as a function of the realization of all the cartel valuations $v_{\mathcal{C}} \equiv (v_j)_{j \in \mathcal{C}}$.

For the cartel to maximize the gains from collusion, it must be the case that whenever the cartel leader wins the auction, i.e. $V_{com} < V_{col}$, the cover bidders drop out at or below V_{com}. In addition, it is reasonable to assume that the cartel members never drop out above their valuations. This way, should the cartel leader renege on its promise to bid up to its valuation and drop out earlier, the cover bidders will not suffer a loss from buying at prices higher than their valuations. We therefore make the following assumption concerning the bidding strategy of a cover bidder. Other than this assumption, a cover bidder's strategy is unrestricted.

Assumption 4. *For any cover bidder $i \in \mathcal{C}$, (i) its dropout price never exceeds its valuation, $B_i^*(v_{\mathcal{C}}, h) \leq v_i$, and (ii) whenever the dropout history h involves the last dropout by the highest competitive bidder, the cover bidder also drops out at that price: $B_i^*(v_{\mathcal{C}}, h) = V_{com}$.*

Since a cover bidder never wins auctions, its actual final bid will be given by $B_i^*(v_{\mathcal{C}}, h^*)$ for the realized history h^* after which it drops out before any other bidder does. The actual bid of a cartel member, as a random variable, is then given by

$$\tilde{B}_i = B_i^*(V_{\mathcal{C}}, h^*)\mathbb{1}\{\ell_i = 0\} + \min\{V_i, V_{com}\}\mathbb{1}\{\ell_i = 1\}.$$

For any bidder $i \in \mathcal{C}$, $V_{com} \leq V_{-i}$ and therefore

$$\min\{V_i, V_{com}\} \leq \min\{V_i, V_{-i}\}, \tag{16}$$

while

$$B_i^*(V_{\mathcal{C}}, h^*) \leq \min\{V_i, V_{-i}\}$$

by (i) and (ii) in Assumption 4. It follows that for cartel members, the counterfactual competitive distribution of bids $G_i^{pred}(b) = \mathbb{P}\{\min\{V_i, V_{-i}\} \le b\}$ weakly stochastically dominates the actual distribution of bids $G_i(b) = \mathbb{P}\{\tilde{B}_i \le b\}$. We show below that stochastic dominance holds in the strict sense.

Proposition 3 (Testable prediction for collusion). *Under Assumptions 1-4, the predicted competitive distribution of i's bids is identified. Moreover, it strictly stochastically dominates the distribution of i's bids if bidder i is collusive:* $G_i(b) \ge G_i^{pred}(b)$ *for all b's, with strict inequalities for some b's.*

Proof. In order to prove that the inequality is strict for some b, by Theorem 1 in Hanoch and Levy (1969),[14] it is sufficient to verify

$$\mathbb{E}[\tilde{B}_i] < \mathbb{E}[\min\{V_i, V_{-i}\}]. \tag{17}$$

For bidder $i \in \mathcal{C}$, define an event $\mathcal{A}_i = \{V_{com} < V_i < V_{col}\}$, and note that $\mathbb{P}(\mathcal{A}_i) > 0$, which holds since the distributions $\{F_j, j \in \mathcal{N}\}$ have the same support.[15] Write $\mathbb{E}[\tilde{B}_i] = \mathbb{E}[\tilde{B}_i \cdot \mathbb{1}(\mathcal{A}_i)] + \mathbb{E}[\tilde{B}_i \cdot \mathbb{1}(\mathcal{A}_i^c)]$. For the first term by Assumption 4, we have $\mathbb{E}[\tilde{B}_i \cdot \mathbb{1}(\mathcal{A}_i)] = \mathbb{E}[V_{com} \cdot \mathbb{1}(\mathcal{A}_i)] < \mathbb{E}[V_i \cdot \mathbb{1}(\mathcal{A}_i)] = \mathbb{E}[\min\{V_i, V_{-i}\} \cdot \mathbb{1}(\mathcal{A}_i)]$. Moreover, (16) implies that $\mathbb{E}[\tilde{B}_i \cdot \mathbb{1}(\mathcal{A}_i^c)] \le \mathbb{E}[\min\{V_i, V_{-i}\} \cdot \mathbb{1}(\mathcal{A}_i^c)]$, and (17) follows.

□

4 Estimation

We consider an i.i.d. sample of L auctions, with each individual auction indexed by $l = 1, ..., L$. For simplicity, we assume that all N bidders participate.

The bids are denoted as b_{il}. For each bidder $i \in \mathcal{N}$, the maximal bid of its rival is denoted as $b_{-il} = \max\{b_{jl} : j \in \mathcal{N} \backslash \{i\}\}$. For $i \in \mathcal{N}$, $w_{il} \in \{0, 1\}$ denotes whether bidder i wins auction l: $w_{il} = 1$ if $b_{il} > b_{-il}$, and $w_{il} = 0$ if $b_{il} < b_{-il}$. In equilibrium, ties will have zero probability, so the allocation rule adopted for tied bids is immaterial. Conditional on losing, i.e. on $w_{il} = 0$, the bidder's valuation v_{il} is revealed and equal to

[14] This theorem states that if $\int H(b)dG_0(b) \ge \int H(b)dG(b)$ for any non-decreasing function $H(\cdot)$, with a strict inequality for at least one such function, then $G_0(b) < G(b)$ for some b. In our case, we pick the identity function, $H(b) = b$, which leads to the comparison of the expected values.

[15] We use the notation $\mathbb{1}(\mathcal{A})$ for the indicator function of an event $\mathcal{A}$.

its bid, while for a winning bid, it is only known that the valuation is at or above the bid: $v_{il} = b_{il}$ if $w_{il} = 0$, and $v_{il} \geq b_{il}$ if $w_{il} = 1$.

Our estimation strategy will be based on the plug-in approach, where the distributions appearing in the decensoring formulae are replaced by their empirical analogues. The distributions G_i^0, G_i can be consistently estimated as

$$\hat{G}_i(b) = \frac{1}{L}\sum_{l=1}^{L} \mathbb{1}(b_{il} \leq b), \qquad \hat{G}_i^0(b) = \frac{1}{L}\sum_{l=1}^{L} \mathbb{1}(b_{il} \leq b, w_{il} = 0). \tag{18}$$

Plugging these estimators into (3), we obtain an estimator for the distribution of valuations of a competitive bidder i:

$$\hat{F}_i(v) = \psi(\hat{G}_i, \hat{G}_i^0)(v). \tag{19}$$

It can be shown, as an application of the Continuous Mapping Theorem, that the estimator $\hat{F}_i$ is consistent on the entire support $[0, \overline{v}]$. The rate of convergence can also be established by standard methods. However, we do not pursue this, as weak convergence results and the bootstrap approach discussed below will be our main tools for inference and testing.

Our main tool for deriving the asymptotic distributions of the estimators and their bootstrap approximations will be the Functional Delta Method (FDM).[16] Using the definition of the functional ψ in (4), its functional derivative, at $H_1 = G_i^0$ and $H_2 = G_i$, can be computed as

$$\psi'(h_1, h_2)(v) = (1 - F_i(v)) \left(\int_0^v \frac{dh_1(u)}{1 - G_i(u)} + \int_0^v \frac{h_2(u) dG_i^0(u)}{(1 - G_i(u))^2} \right). \tag{20}$$

Standard results for weak convergence of empirical processes imply, jointly for all i's,

$$\sqrt{L}\big(\hat{G}_i - G_i, \hat{G}_i^0 - G_i^0\big) \rightsquigarrow \big(\mathbb{G}_i, \mathbb{G}_i^0\big), \tag{21}$$

where $\rightsquigarrow$ denotes weak convergence, and $\mathbb{G}_i$ and $\mathbb{G}_i^0$ are (correlated) tight mean-zero Gaussian processes on $[0, \overline{v}]$.[17] The covariance functions of these processes can be com-

[16] See e.g. Chapter 20 of van der Vaart (1998).
[17] See also Lemma B.3 in the Appendix.

puted as

$$\begin{aligned}
\mathbb{E}\mathbb{G}_i(v_1)\mathbb{G}_i(v_2) &= G_i(v_1 \wedge v_2) - G_i(v_1)G_i(v_2), \\
\mathbb{E}\mathbb{G}_i^0(v_1)\mathbb{G}_i^0(v_2) &= G_i^0(v_1 \wedge v_2) - G_i^0(v_1)G_i^0(v_2), \quad \text{and} \\
\mathbb{E}\mathbb{G}_i(v_1)\mathbb{G}_i^0(v_2) &= G_i^0(v_1 \wedge v_2) - G_i(v_1)G_i^0(v_2).
\end{aligned} \tag{22}$$

Consider any proper sub-interval $[0, \overline{v}_0] \subset [0, \overline{v})$. The functional ψ can be shown to be Hadamard differentiable on the space of bounded, right-continuous, left-limit (cadlag) functions on $[0, \overline{v}_0]$ (with the derivative given by (20)). The FDM then implies weak convergence of the process $\sqrt{L}(\hat{F}_i(v) - F_i(v))$, to a tight Gaussian process on $[0, \overline{v}_0]$,

$$\begin{aligned}
\sqrt{L}(\hat{F}_i(v) - F_i(v)) &\rightsquigarrow \psi'(\mathbb{G}_i, \mathbb{G}_i^0)(v) \\
&= (1 - F_i(v)) \left(\int_0^v \frac{d\mathbb{G}_i^0(u)}{1 - G_i(u)} + \int_0^v \frac{\mathbb{G}_i(u) dG_i^0(u)}{(1 - G_i(u))^2} \right).
\end{aligned} \tag{23}$$

The estimator $\hat{F}_i$, together with some other estimators defined later using ψ, will be used as inputs for construction of estimators using the de-censoring formula under collusion in (12). Because in (12) the integral under the exponent extends up to the upper boundary of the support $\overline{v}$, this requires that the input estimators weakly converge on the entire support $[0, \overline{v}]$. However, the main difficulty in obtaining such results is that the denominator $1 - G_i(u)$ in (3) tends to 0 as u approaches $\overline{v}$, and consequently, the functional ψ is not Hadamard differentiable on the space of functions defined on the entire support $[0, \overline{v}]$.

In order to overcome this difficulty, we propose a trimmed version of the estimator. The trimmed estimator is denoted as $\tilde{F}_i(v)$ and is defined as

$$\tilde{F}_i(v) \equiv \hat{F}_i(v \wedge \bar{v}_{i,L}),$$

where $\bar{v}_{i,L} \uparrow \overline{v}$ is the trimming sequence, and the convergence of $\bar{v}_{i,L}$ is in probability. We define $\bar{v}_{i,L}$ through the quantile transformation $\hat{G}_i^{-1}(t_L)$,[18] where $t_L \uparrow 1$:

$$\bar{v}_{i,L} \equiv \hat{G}_i^{-1}(t_L).$$

[18] We use the standard definition of quantile transformations: For a CDF H, $H^{-1}(t) = \inf\{v : H(v) \geq t\}$, where $t \in (0, 1)$. In fact, since we considering distributions with compact supports, $(0, 1)$ can be changed to $[0, 1]$.

In other words, we trim values v using a sequence of extreme quantiles of the estimated distribution of bids. Such a trimming scheme is convenient as it does not require estimation of the upper bound of the support of the distribution of valuations. The trimming parameter $\bar{v}_{i,L}$ has to approach the upper bound of the support at a rate faster than $L^{-1/2}$ to avoid an asymptotic bias. At the same time the rate has to be sufficiently slow to (uniformly) control the approximation error in the FDM. The assumption below prescribes sufficient bounds on the rate.

Assumption 5. *The trimming parameter t_L satisfies $t_L = 1 - L^{-\beta}$ with $1/2 < \beta < 3/4$.*

We also make the following smoothness assumption.

Assumption 6. *The CDFs F_i's have densities f_i's, which are smooth (belong to C^∞) and bounded away from zero on the support $[0, \bar{v}]$.*

With these assumptions, the result in (23) can be strengthened to hold over the entire support $[0, \overline{v}]$.

Proposition 4 (Weak convergence under competition)**.** *Under Assumptions 1–6, the following weak convergence holds for the trimmed estimators $\tilde{F}_i$ jointly for all i, over the entire support $[0, \overline{v}]$,*

$$\sqrt{L}(\tilde{F}_i - F_i) \rightsquigarrow \psi'(\mathbb{G}_i, \mathbb{G}_i^0).$$

We now turn to estimation of the distribution of bidder valuations under collusion. Our estimation strategy again follows the plug-in approach. It is convenient to define the expression appearing on the right-hand side of collusion de-censoring formula (12) as a functional:

$$\psi_{col}(H_1, H_2)(v) \equiv \exp\left(- \int_v^\infty \frac{dH_1(u)}{H_2(u)} \right).$$

The identification result in Proposition 2 can now be stated as a functional of $F_i^\ell(\cdot)$ and $F_{col}(\cdot)$:

$$F_i(v) = \psi_{col}(F_i^\ell, F_{col})(v),$$

where $F_i^\ell \equiv \psi(G_{col}^{0,\ell}, G_i^\ell)$ and $F_{col} \equiv \psi(G_{col}^0, G_{col})$.

The distributions F_i^ℓ and F_{col} are estimated as follows. First, we estimate the distributions G_i^ℓ and $G_i^{0,\ell}$ as the empirical averages in parallel to (18), however, conditional on the event that i is the leader, $\ell_i = 1$:

$$\hat{G}_i^{\ell}(b) = \frac{\sum_{l=1}^{L} \mathbb{1}(b_{il} \leq b, \ell_{il} = 1)}{\sum_{l=1}^{L} \mathbb{1}(\ell_{il} = 1)}, \qquad \hat{G}_i^{0,\ell}(b) = \frac{\sum_{l=1}^{L} \mathbb{1}(b_{il} \leq b, w_{il} = 0, \ell_{il} = 1)}{\sum_{l=1}^{L} \mathbb{1}(\ell_{il} = 1)}. \quad (24)$$

We similarly estimate the distributions for the maximum bid $b_l^* \equiv \max_{i \in \mathcal{N}_{col}} b_{il}$ in the group of suspects $\mathcal{N}_{col}$:

$$\hat{G}_{col}(b) = \frac{1}{L} \sum_{l=1}^{L} \mathbb{1}(b_l^* \leq b), \qquad \hat{G}_{col}^0(b) = \frac{1}{L} \sum_{l=1}^{L} \mathbb{1}(b_l^* \leq b, w_l = 0). \quad (25)$$

These estimators are then plugged in to obtain the consistent estimators $\hat{F}_{i,L}^{\ell}$ and $\hat{F}_{col}$:

$$\hat{F}_i^{\ell} = \psi(\hat{G}_i^{0,\ell}, \hat{G}_{i,L}^{\ell}), \qquad \hat{F}_{col} = \psi(\hat{G}_{col}^0, \hat{G}_{col}). \quad (26)$$

Using the trimmed estimators

$$\tilde{F}_i^{\ell}(v) \equiv \hat{F}_i^{\ell}(v \wedge \bar{v}_{i,L}), \qquad \tilde{F}_{col} \equiv \hat{F}_{col}(v \wedge \bar{v}_{col,L}), \quad (27)$$

where $\bar{v}_{col,L} \equiv \hat{G}_{col}^{-1}(t_L)$, the estimator of F_i under collusion is defined by the plug-in approach as

$$\tilde{F}_i^{col} = \psi_{col}(\tilde{F}_i^{\ell}, \tilde{F}_{col}). \quad (28)$$

In parallel to the result in Proposition 4, one can show weak convergence on the entire support $[0, \bar{v}]$ of the empirical processes for $\tilde{F}_i^{\ell}$ and $\tilde{F}_{col}$ to tight Gaussian processes, denoted respectively as $\mathbb{F}_i^{\ell}$ and $\mathbb{F}_{col}$:

$$\sqrt{L}(\tilde{F}_i^{\ell} - F_i^{\ell}) \rightsquigarrow \mathbb{F}_i^{\ell} \equiv \psi'(\mathbb{G}_i^{0,\ell}, \mathbb{G}_i^{\ell}), \qquad \sqrt{L}(\tilde{F}_{col} - F_{col}) \rightsquigarrow \mathbb{F}_{col} \equiv \psi'(\mathbb{G}_{col}^0, \mathbb{G}_{col}), \quad (29)$$

where $\left(\mathbb{G}_i^{0,\ell}, \mathbb{G}_i^{\ell}, \mathbb{G}_{col}^0, \mathbb{G}_{col}\right)$ are (correlated) Gaussian processes that arise in the weak convergence of the corresponding estimators:

$$\sqrt{L}(\hat{G}_i^{0,\ell} - G_i^{0,\ell}, \hat{G}_i^{\ell} - G_i^{\ell}, \hat{G}_{col}^0 - G_{col}^0, \hat{G}_{col} - G_{col}) \rightsquigarrow \left(\mathbb{G}_i^{0,\ell}, \mathbb{G}_i^{\ell}, \mathbb{G}_{col}^0, \mathbb{G}_{col}\right), \quad (30)$$

and the weak convergence holds jointly with that in (21) and across i's. The corresponding covariances are defined similarly to those in (22). The functional derivative of ψ_{col}, at $H_1 = F_i^\ell$ and $H_2 = F_{col}$, can be computed as

$$\psi'_{col}(h_1, h_2)(v) = F_i(v) \left(- \int_v^{\overline{v}} \frac{dh_1(u)}{F_{col}(u)} + \int_v^{\overline{v}} \frac{h_2(u) dF_i^\ell(u)}{F_{col}^2(u)} \right).$$

The following proposition establishes a result analogous to that in Proposition 4, but under collusion.

Proposition 5 (Weak convergence under collusion). *Under Assumptions 1–6, the following weak convergence holds jointly for all i's, over any proper sub-interval* $[\underline{v}_0, \overline{v}] \subset (0, \overline{v}]$.

$$\sqrt{L}(\tilde{F}_i^{col} - F_i) \rightsquigarrow \psi'_{col}(\mathbb{F}_i^\ell, \mathbb{F}_{col}),$$

where $\mathbb{F}_i^\ell$ *and* $\mathbb{F}_{col}$ *are defined in* (29).

Remark 1. The weak convergence in Proposition 5 is over any compact interval that excludes 0, the lower boundary of the support. The reason for this is that $F_{col}(u) \to 0$ as $u \downarrow 0$, which creates a "small denominator" problem: the functional ψ_{col} is not Hadamard differentiable on the space of functions defined on the entire support $[0, \overline{v}]$. However, it is Hadamard differentiable on any sub-interval with a strictly positive lower bound. This is the same difficulty encountered for the estimator $\hat{F}_i$ under competition, which we resolved by trimming the support of valuations from above. We conjecture that a similar trimming approach, now from below, would work here as well, but we do not pursue such an extension.

In finite samples, it is unlikely to observe a cartel leader with a very small valuation. Therefore, the estimator $\hat{F}_i^{col}$ will suffer from a substantial small sample bias for valuations v near zero. Thus, extending Proposition 5 to the lower bound of the support is not practical.

Similarly, one can expect a substantial small sample bias for valuations v near $\bar{v}$: in finite samples, it is unlikely to observe a cartel leader with a very large valuation near the upper boundary of the support losing to the competitive fringe. Hence, for testing purposes, we will focus below on proper sub-intervals $[\underline{v}_0, \overline{v}_0] \subset (0, \overline{v})$.

4.1 Econometric test of collusion

We begin by testing the null hypothesis that bidder i bids competitively. The null can be stated as $H_{0,i} : G_i(b) = G_i^{pred}(b)$ for all b. The corresponding alternative hypothesis is collusive behavior of bidder i, which can be stated as $H_{1,i} : G_i(b) \geq G_i^{pred}(b)$ with strict inequalities for some b's.

The basis of the test will be the deviation of the actual CDF of bids submitted in the auction $G_i(b)$ from the predicted competitive CDF of i's bids $G_i^{pred}(b)$. Pick a compact proper sub-interval $[\underline{v}_0, \overline{v}_0] \subset (0, \overline{v})$, and consider a maximum deviation statistic

$$\hat{T}_i = \max_{b \in [\underline{v}_0, \overline{v}_0]} \left[\hat{\Delta}_i(b) \right]_+, \tag{31}$$

where

$$\hat{\Delta}_i(b) \equiv \hat{G}_i(b) - \hat{G}_i^{pred}(b)$$

denotes the difference between the estimated distribution of bids of bidder i and the estimated predicted distribution of bids for bidder i under competition, and

$$[x]_+ = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Large values of this statistic will be indicative of collusion.

Using (13) and (15), we can express the predicted (or counterfactual) CDF of bids for *suspect* bidder i under competition as a functional

$$\begin{aligned} G_i^{pred} =& \psi_{i,pred}\big(F_i, \{F_j\}_{j \in \mathcal{N}_{col} \setminus \{i\}}, \{F_j\}_{j \in \mathcal{N}_{com}}\big) \\ \equiv& 1 - (1 - F_i) \left(1 - \prod_{j \in \mathcal{N}_{col} \setminus \{i\}} F_j \prod_{j \in \mathcal{N}_{com}} F_j \right). \end{aligned} \tag{32}$$

The functional $\psi_{i,pred}$ involves only products of CDFs and, consequently, is Hadamard differentiable. We denote its Hadamard derivative by $\psi'_{i,pred}\big(h_i, \{h_j\}_{j \in \mathcal{N}_{col} \setminus \{i\}}, \{h_j\}_{j \in \mathcal{N}_{com}}\big)$. Note that for $j \in \mathcal{N}_{col}$, $F_j = \psi_{col}(F_j^{\ell}, F_{col})$. Similarly for $j \in \mathcal{N}_{com}$, $F_j = \psi(G_j^0, G_j)$. Therefore, under the null of competition, a repeated application of the FDM together with Propositions 4 and 5 implies that the difference between the estimated distributions $\hat{G}_i$ and $\hat{G}_i^{pred}$ converges weakly to a mean-zero Gaussian process on $[\underline{v}_0, \overline{v}_0]$:

$$\sqrt{L}\hat{\Delta}_i(b) = \sqrt{L}(\hat{G}_i - \hat{G}_i^{pred}) \rightsquigarrow \mathbb{G}_i - \mathbb{G}_i^{pred},$$

where

$$\mathbb{G}_i^{pred} = \psi'_{i,pred}\left(\psi'_{col}(\mathbb{F}_i^{\ell}, \mathbb{F}_{col}), \left\{\psi'_{col}(\mathbb{F}_j^{\ell}, \mathbb{F}_{col})\right\}_{j\in\mathcal{N}_{col}\setminus\{i\}}, \left\{\psi'(\mathbb{G}_j^0, \mathbb{G}_j)\right\}_{j\in\mathcal{N}_{com}}\right). \tag{33}$$

The Continuous Mapping Theorem then implies that under the null of competition, the statistic $\sqrt{L}\hat{T}_i$ also converges weakly:

$$\sqrt{L}\hat{T}_i \rightsquigarrow \max_{b\in[\underline{v}_0,\overline{v}_0]}[\mathbb{G}_i(b) - \mathbb{G}_i^{pred}(b)]_+. \tag{34}$$

At the same time according to Assumption 4, the statistic $\sqrt{L}\hat{T}_i$ is divergent if bidder i participates in the cartel.

In principle, the limiting distribution of $\sqrt{L}\hat{T}_i$ that appears above could be computed through the simulation of the Gaussian processes $\mathbb{G}_i(b)$ and $\mathbb{G}_i^{pred}(b)$. However, since the covariance structure of the limiting process is complicated due to the multi-step nature of our estimator, we propose to approximate the null distribution of our test statistic by the bootstrap.

The bootstrap samples are generated by drawing randomly with replacement L auctions from the original sample of L auctions. Let $\{(b_{1l}^{\dagger}, \ldots, b_{Nl}^{\dagger}) : l = 1, \ldots, L\}$ be a bootstrap sample, and M be the number of bootstrap samples. In each bootstrap sample, we construct $\hat{G}_i^{\dagger}$ and $\hat{G}_i^{0,\dagger}$, which are the bootstrap analogues of $\hat{G}_i$ and $\hat{G}_i^0$ respectively. The bootstrap version of the trimmed estimator $\tilde{F}_i$ is given by

$$\tilde{F}_i^{\dagger}(v) = \psi(\hat{G}_i^{0,\dagger}, \hat{G}_i^{\dagger})(v \wedge \bar{v}_{i,L}^{\dagger}),$$

where $\bar{v}_{i,L}^{\dagger} \equiv (\hat{G}_i^{\dagger})^{-1}(t_L)$, and the trimming parameter t_L is defined in Assumption 5.

We can similarly define the bootstrap estimators corresponding to the decensoring formula under collusion. Our functional notation allows to define those estimators conveniently as follows. Let $\hat{G}_i^{\ell,\dagger}$, $\hat{G}_i^{0,\ell,\dagger}$, $\hat{G}_{col}^{\dagger}$, and $\hat{G}_{col}^{0,\dagger}$ be the bootstrap analogues of $\hat{G}_i^{\ell}$, $\hat{G}_i^{0,\ell}$, $\hat{G}_{col}$, and $\hat{G}_{col}^0$ respectively, see equations (24) and (25). As in equations (26) and (27), we have $\tilde{F}_i^{\ell,\dagger}(v) = \psi(\hat{G}_i^{0,\ell,\dagger}, \hat{G}_i^{\ell,\dagger})(v \wedge \bar{v}_{i,L}^{\dagger})$, and $\tilde{F}_{col}^{\dagger}(v) = \psi(\hat{G}_{col}^{0,\dagger}, \hat{G}_{col}^{\dagger})(v \wedge \bar{v}_{col,L}^{\dagger})$ with $\bar{v}_{col,L}^{\dagger} \equiv (\hat{G}_{col}^{\dagger})^{-1}(t_L)$. Moreover, following equation (28), the bootstrap estimator of the distribution F_i under potential collusion is $\tilde{F}_i^{col,\dagger} = \psi_{col}(\tilde{F}_i^{\ell,\dagger}, \tilde{F}_{col}^{\dagger})$. We can now define the bootstrap analogue of the counterfactual (predicted) distribution of bids of bidder i:

$$\hat{G}_i^{pred,\dagger} = \psi_{i,pred}\big(\tilde{F}_i^{col,\dagger}, \{\tilde{F}_j^{col,\dagger}\}_{j\in\mathcal{N}_{col}\setminus\{i\}}, \{\tilde{F}_j^{\dagger}\}_{j\in\mathcal{N}_{com}}\big).$$

Lastly, we construct the bootstrap analogue of $\hat{T}_i$:

$$\hat{T}_i^\dagger = \max_{b \in [\underline{v}_0, \overline{v}_0]} \left[\hat{\Delta}_i^\dagger(b) - \hat{\Delta}_i(b) \right]_+,$$

where

$$\hat{\Delta}_i^\dagger(b) = \hat{G}_i^\dagger(b) - \hat{G}_i^{pred,\dagger}(b)$$

is the bootstrap analogue of $\hat{\Delta}_i(b)$.[19]

Let $\{\hat{T}_{i,m}^\dagger : m = 1, \ldots M\}$ be the collection of the bootstrap test statistics computed in bootstrap samples 1 through M. The critical value $\hat{c}_{i,1-\alpha}$ is the $(1-\alpha)$-th sample quantile of $\{\hat{T}_{i,m}^\dagger : m = 1, \ldots M\}$, where α is the desired asymptotic significance level. The null hypothesis of competitive behaviour for bidder i is rejected when $\hat{T}_i > \hat{c}_{i,1-\alpha}$.

Our next proposition establishes the validity of the bootstrap procedures.

Proposition 6. *Under Assumptions 1–6, the following results hold jointly:*

$$\sqrt{L}(\tilde{F}_i^\dagger - \hat{F}_i) \rightsquigarrow \psi'(\mathbb{G}_i, \mathbb{G}_i^0), \qquad v \in [0, \overline{v}], \tag{35}$$

$$\sqrt{L}(\tilde{F}_i^{col,\dagger} - \tilde{F}_i^{col}) \rightsquigarrow \psi'_{col}(\mathbb{F}_i^\ell, \mathbb{F}_{col}), \qquad v \in [\underline{v}_0, \overline{v}], \tag{36}$$

$$\sqrt{L}(\hat{\Delta}_i^\dagger - \hat{\Delta}_i) \rightsquigarrow \mathbb{G}_i - \mathbb{G}_i^{pred}, \qquad b \in [\underline{v}_0, \overline{v}]. \tag{37}$$

Moreover, the results also hold jointly across i's.

Remark 2. The proof of Proposition 6 relies on the strong approximation results for the bootstrap in Chen and Lo (1997). The Gaussian processes $\mathbb{G}_i$, $\mathbb{G}_i^0$, $\mathbb{F}_i^\ell$, $\mathbb{F}_{col}$, and $\mathbb{G}_i^{pred}$ in Proposition 6 should be viewed as independent copies of the corresponding processes appearing in Propositions 4, 5, and equation (33).

The validity of the bootstrap test now follows from (37) as an application of the Continuous Mapping Theorem.

Corollary 1. *Under Assumptions 1–6,*

$$\sqrt{L}\hat{T}_i^\dagger \rightsquigarrow \max_{b \in [\underline{v}_0, \overline{v}_0]} [\mathbb{G}_i(b) - \mathbb{G}_i^{pred}(b)]_+. \tag{38}$$

[19]Note that to ensure a valid bootstrap approximation, we must re-center $\hat{\Delta}_i^\dagger(b)$ by $\hat{\Delta}_i(b)$. The re-centering is needed to ensure that the bootstrap version of the test statistic is generated under the null.

Remark 3. Consistency of the bootstrap testing procedure follows from (34) and (38) by Polýa's Theorem, i.e. $\mathbb{P}(\sqrt{L}\hat{T}_i > \hat{c}_{i,1-\alpha}) \to \alpha$ when $H_{0,i} : G_i(b) = G_i^{pred}(b)$ is true.

Our collusion test can be applied bidder by bidder to construct an estimated set of colluders (a cartel set). However, due to the multiple hypothesis nature of this procedure, it is necessary to control the overall probability of falsely implicating a competitive firm. This can be achieved, for example, by using the Holm-Bonferroni sequential testing procedure that we now describe. Let α denote the overall significance level. The procedure is performed by ordering the individual p-values from smallest to largest,

$$p_{(1)} \leq \dots \leq p_{(K)},$$

where K is the number of suspects, i.e. the number of bidders in $\mathcal{N}_{col}$.

Step 1 The firm with the smallest p-value is included in the cartel set if

$$p_{(1)} < \alpha/K,$$

after which one proceeds to Step 2. Otherwise the procedure stops and none of the firms are included in the cartel.

Step 2 The firm with the second-smallest p-value is tested next. It is included in the cartel if

$$p_{(2)} < \alpha/(K-1),$$

after which one proceeds to the next step. Otherwise the procedure stops and none of the firms are included in the cartel. (The first firm that was included is now excluded as there can never be a single-firm cartel.)

Step 3 The firm with the third-lowest p-value is tested and is included in the cartel if

$$p_{(3)} < \alpha/(K-3),$$

after which one proceeds to the next step. Otherwise, the procedure stops with the two-firm cartel (firms 1 and 2).

And so on until termination.

Once the composition of the cartel $\mathcal{C}$ has been estimated, we can investigate the damage caused by collusion. The predicted auction price under competition is distributed as the second-order statistic:

$$G^{pred}(p) \equiv \sum_{j \in \mathcal{N}} \prod_{i \in \mathcal{N} \setminus \{j\}} F_i(p)(1 - F_j(p)) + \prod_{i \in \mathcal{N}} F_i(p).$$

This distribution can be estimated by the plug-in approach using the estimates of $F_i(p)$ under competition for $i \in \mathcal{N} \setminus \hat{\mathcal{C}}$, and the estimates under collusion for $i \in \hat{\mathcal{C}}$ under collusion, where $\hat{\mathcal{C}}$ denotes the estimated cartel set.

Remark 4 (Heterogeneity)**.** We have focused on the case where the same object is auctioned. In many applications, auction-specific heterogeneity is important. Following Haile et al. (2003), the standard approach in the literature is to control for heterogeneity through a first-step regression,

$$b_{il} = m(x_l; \theta) + \varepsilon_{il},$$

where the error terms ε_{il} are independent of the object characteristics x_l (and are also independent across bidders). This regression can be estimated parametrically as in Haile et al. (2003). Our estimators can be applied to the homogenized bids $\hat{\varepsilon}_{il}$ resulting from this regression, and our bootstrap test of collusion can be similarly performed with the homogenized bids.

5 Monte Carlo experiment

In this section, we investigate the small-sample performance of our individual test in a Monte Carlo experiment. We consider a setting with 3 bidders who draw values independently from the same distribution, specified as log-normal, $\log V_i \sim N(0, 1)$. Bidder 1 is always competitive, while bidders 2 and 3 may collude. We assume that collusion takes the following form: bidders 2 and 3 are aware of the presence of the competitive bidder, and do not compete with each other if the competitive bidder has dropped out. Thus, if the maximal cartel valuation $\max\{V_2, V_3\} > V_1$, the bidding stops at the price equal to the competitive bidder's valuation V_1 even if $\min\{V_2, V_3\} > V_1$ and the price under competition would be V_2. Otherwise, if $\max\{V_2, V_3\} \le V_1$, then the

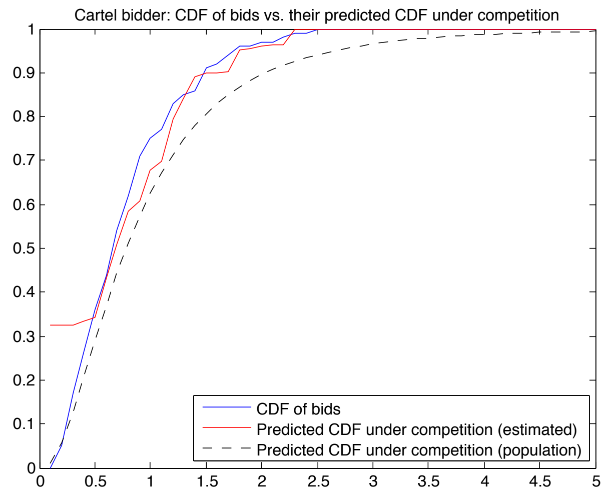

Figure 1: Suspect cartel bidder; the data are generated under collusion. The sample size is 100 auctions.

competitive bidder wins the auction at the price equal to the cartel leader's valuation $\max\{V_2, V_3\}$.

The estimated predicted competitive distribution when the data are generated under *collusion* is reported in Figures 1 and 2. All figures contain the plots of the estimated actual bid distribution, the true predicted competitive bid distribution, and the estimated predicted competitive bid distributions. For the smaller sample size $L = 100$, both small sample bias and sample variation are clearly present. Still, even though the estimated predicted bid distribution is not too close to the true one, for most values it is below the actual bid distribution (i.e. shifted towards higher bids). This suggests that even in small samples, collusion might be detectable. The situation improves dramatically for the larger sample, $L = 400$ auctions. Indeed, it is remarkable how close the estimated predicted distribution is to the true population distribution. If the data instead are generated under competition, then the three curves are very close to each other for the sample of $L = 400$ auctions; see Figure 3.

To evaluate size properties of our testing procedure, we simulated bids data under competition, i.e. for all three bidders their bids are generated as

$$B_i = \min\{V_i, \max_{j \neq i}\{V_j\}\}, \quad i = 1, 2, 3.$$

However, when applying the de-censoring formulas and computing the test statistics in

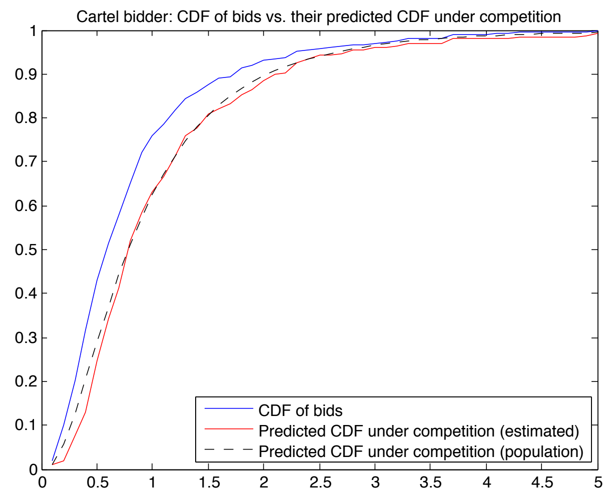

Figure 2: Suspect cartel bidder; the data are generated under collusion. The sample size is 400 auctions.

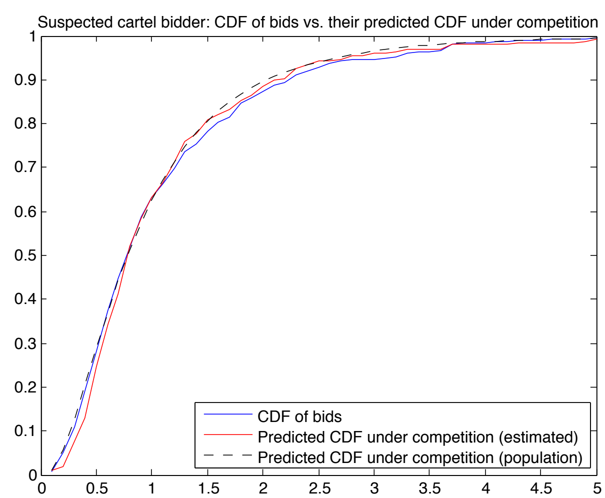

Figure 3: Suspect cartel bidder; the data are generated under competition. The sample size is 400 auctions.

Table 1: Average rejection rates of the bootstrap test for collusion for different significance levels and sample sizes (L)

significance level	$L = 100$	$L = 400$	$L = 100$	$L = 400$
	Competition (H_0)		Collusion (H_1)	
0.01	0.009	0.008	0.403	0.934
0.05	0.030	0.043	0.626	0.981
0.10	0.067	0.080	0.732	0.994

the original and bootstrap samples, we proceeded under the assumption that bidders 2 and 3 were collusive. We expect that in this case there should not be any significant differences between the CDF of bids for a suspected cartel member ($\hat{G}$) and the predicted CDF of bids under competition ($\hat{G}^{pred}$).

For power computations, bids for cartel members (bidders 2 and 3) were generated as described in the beginning of the section:

$$B_i = \min\{V_i, V_1\}, \quad i = 2, 3.$$

In this case, we expect to see the CDF of bids for a suspected cartel member ($\hat{G}$) to be positioned *above* the predicted CDF of bids under competition ($\hat{G}^{pred}$), i.e. our test should reject the null of competitive behaviour for bidders 2 and 3 with high probability.

The results of our Monte Carlo study are summarized in Table 1. The table reports average rejection rates for 1,000 Monte Carlo repetitions. To compute bootstrap critical values, we used 1,000 bootstrap samples (at each Monte Carlo replication).

The test is slightly undersized in small samples of 100 auctions. However, in moderate size samples of 400 auctions, the rejection rates under the null of competitive behaviour are very close to the nominal levels. The test also has very good power properties. For example in the case of collusive behaviour for bidders 2 and 3, the 5% test rejects the null with probabilities exceeding 60% in small samples and 98% in moderate samples.[20]

[20]The test was performed for bidder 2.

6 Concluding remarks

The research in this paper can be extended in a number of directions. Below, we discuss three important but challenging extensions.

First, we restrict attention to English auctions. Can our approach be extended to another popular format, first-price auctions (FPA)? In English auctions, bidders stay in the auction up to their valuations. As we have shown, this crucial feature allows one to identify the the distribution of valuations of a given bidder *regardless* of whether other bidders are colluding and who participates in the cartel. In FPAs, bidders bid less than their values, and the competitive bids depend on whether there is a cartel, and on the cartel composition. A combination of our approach with the identification and estimation methodology for first-price auctions proposed in Guerre et al. (2000) is clearly desirable.

The second extension concerns relaxation of the efficient cartel hypothesis. While many papers in the empirical auction literature assume efficient collusion, this is obviously a limitation. As Asker (2010) has demonstrated for a postal stamp cartel, a cartel large enough to exercise market power may include bidders that are quite different, and may adopt a knockout auction that leads to inefficient allocation. If the form of the knockout auction is known to the researcher, one could use this information to extend our approach. This extension is left for future research.

Third, our approach relies on the button model of the English auction, which as we have argued, is applicable to recent Internet auction designs with minimal information disclosure, where bidders only see the status of their bid (winning or losing). In particular, the model is suitable for our empirical application. In this model, it is a dominant strategy for a bidder to drop out at its valuation. Haile and Tamer (2003) argue that this assumption is unrealistic in traditional English auctions and develop sharp nonparametric bounds on the distributions of valuations when it does not hold. Whether or not their bounding approach could be extended to collusion is an open question also left for future research.

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A Appendix: Extended Functional Delta Method

The following lemma is an extension of the FDM (van der Vaart, 1998, Theorem 20.8) and allows for functionals that depend on the sample size L. This includes functionals with sample-size-dependent trimming.

Lemma A.1 (Extended Functional Delta Method). *Let $\mathbb{D}$ and $\mathbb{E}$ be normed linear spaces. Suppose that:*

(i) $r_L\|\phi_L(F) - \phi(F)\| \to 0$, *where* $r_L \to \infty$ *as* $L \to \infty$, *and* $\phi_L, \phi : \mathbb{D} \to \mathbb{E}$.

(ii) There is a continuous linear map $\phi'_{F,L} : \mathbb{D} \to \mathbb{E}$ *such that, for every compact* $D \in \mathbb{D}_0 \subset \mathbb{D}$,

$$\sup_{h \in D : F + h/r_L \in \mathbb{D}} \left\| \frac{\phi_L(F + h/r_L) - \phi_L(F)}{1/r_L} - \phi'_{F,L}(h) \right\| \to 0.$$

(iii) $\|\phi'_{F,L}(h_L) - \phi'_F(h)\| \to 0$ *for all* h_L *such that* $\|h_L - h\| \to 0$ *with* $h \in \mathbb{D}_0$, *where* $\phi'_F : \mathbb{D}_0 \to \mathbb{E}$ *is a continuous linear map.*

(iv) $\mathbb{G}_L = r_L(F_L - F) \rightsquigarrow \mathbb{G}$, *where* $P(\mathbb{G} \in \mathbb{D}_0) = 1$.

Then, $r_L(\phi_L(F_L) - \phi(F)) \rightsquigarrow \phi'_F(\mathbb{G})$.

Proof. First, $r_L(\phi_L(F_L) - \phi(F)) = r_L(\phi_L(F_L) - \phi_L(F)) + r_L(\phi_L(F) - \phi(F))$, where the second term is $o(1)$ by Condition (i) of the Lemma. Next, $r_L(\phi_L(F_L) - \phi_L(F)) = r_L(\phi_L(F + \mathbb{G}_L/r_L) - \phi_L(F)) = (\phi_L(F + \mathbb{G}_L/r_L) - \phi_L(F))/(1/r_L) - \phi'_{F,L}(\mathbb{G}_L) + \phi'_{F,L}(\mathbb{G}_L) = o_p(1) + \phi'_{F,L}(\mathbb{G}_L)$, where the last equality is by (ii), and the $o_p(1)$ term converges in outer probability. The result now follows by (iii), (iv) and the Extended Continuous Mapping Theorem (van der Vaart, 1998, Theorem 18.11(i)).

□

B Appendix: Proofs of the main results

For the reasons that will be explained shortly, it will prove convenient to re-state our decensoring formulas using quantile transformations. For a CDF function $G(\cdot)$, let $G^{-1}(\tau)$

denote its quantile function, $\tau \in (0,1)$. We introduce the following additional notation. Given a value v, we define

$$
\begin{aligned}
t &= G_i(v), \\
S_i(t) &= F_i(G_i^{-1}(t)) \qquad (39)\\
\implies F_i(v) &= S_i(G_i(v)). \qquad (40)
\end{aligned}
$$

In addition, we define the following quantile transformation of $G_i^0(v) = P(B_{il} \leq v, w_{il} = 0)$:

$$
\mu_i(\tau) = G_i^0(G_i^{-1}(\tau)). \qquad (41)
$$

Using those definitions, equation (3) implies the following expression for the quantile transformation $S_i(t)$:

$$
S_i(t) = 1 - \exp\left(-\int_0^t \frac{d\mu_i(\tau)}{1-\tau}\right). \qquad (42)
$$

The estimated version of $S_i(t)$ can be stated analogously. With $\hat{G}_i$ and $\hat{G}_i^0$ denoting the estimated versions G_i and G_i^0 respectively, we define $\hat{\mu}_i(\tau) = \hat{G}_i^0(\hat{G}_i^{-1}(\tau))$. We have now

$$
\hat{S}_i(t) = 1 - \exp\left(-\int_0^t \frac{d\hat{\mu}_i(\tau)}{1-\tau}\right),
$$

where $\hat{S}_i$ is the estimated version of S_i. Thus, our quantile transformation eliminates the random denominator in the integral expression for the estimated CDF. Note that the estimator $\hat{F}_i(v)$ in (19) can be equivalently written via (40), as $\hat{F}_i(v) = \hat{S}_i(\hat{G}_i(v))$. Moreover, one can define the trimmed version of the estimator $\hat{S}_i(t)$, where in view of Assumption 5, the trimming is applied using the sequence t_L:

$$
\begin{aligned}
\tilde{S}_i(t) &= \hat{S}_i(t \wedge t_L) \\
&= \hat{F}_i(\hat{G}_i^{-1}(t) \wedge \hat{G}_i^{-1}(t_L)) \\
&= \hat{F}_i(\hat{G}_i^{-1}(t \wedge t_L)) \\
&= 1 - \exp\left(-\int_0^{t \wedge t_L} \frac{d\hat{\mu}_i(\tau)}{1-\tau}\right).
\end{aligned}
$$

The following notion of continuity plays an important role in the proofs:

Definition 1. *A real-valued function h is α-Hölder continuous, denoted $h \in \mathscr{H}_\alpha$, if there are constants $C > 0$ and $\alpha > 0$ such that $|h(x) - h(y)| \leq C|x - y|^\alpha$ for all x and y in the domain of h.*

The following lemma shows that the derivative of the measure μ_i is α-Hölder continuous with $\alpha = 1/2$.

Lemma B.2. *Suppose that Assumption 1 holds. The function*

$$\mu_i'(t) = \frac{g_i^0(G_i^{-1}(t))}{g_i(G_i^{-1}(t))}$$

is bounded from above and away from zero, continuously differentiable on $[0, 1)$, and α-Hölder continuous at $t = 1$ with $\alpha = 1/2$.

Proof of Lemma B.2. It is convenient to write

$$\mu_i'(t) = r_i(G_i^{-1}(t)),$$

where

$$r_i(v) \equiv \frac{g_i^0(v)}{g_i(v)}. \tag{43}$$

We first show that $r_i(\cdot)$ is continuously differentiable on the entire support $[0, \overline{v}]$, including the upper boundary $\overline{v}$. We have

$$\begin{aligned} r_i(v) &= \frac{f_i(v)(1 - F_{-i}(v))}{f_i(v)(1 - F_{-i}(v)) + f_{-i}(v)(1 - F_i(v))} \\ &= \frac{f_i(v)\frac{1 - F_{-i}(v)}{\overline{v} - v}}{f_i(v)\frac{1 - F_{-i}(v)}{\overline{v} - v} + f_{-i}(v)\frac{1 - F_i(v)}{\overline{v} - v}} \\ &= \frac{f_i(v)h_{-i}(v)}{f_i(v)h_{-i}(v) + f_{-i}(v)h_i(v)}, \end{aligned}$$

where we denoted

$$h_i(v) = \frac{1 - F_i(v)}{\overline{v} - v}, \qquad h_{-i}(v) = \frac{1 - F_{-i}(v)}{\overline{v} - v}.$$

Our assumption that the distributions $F_i(\cdot)$ have densities $f_i(\cdot)$, smooth (C^∞) and bounded away from 0 on the support $[0, \overline{v}]$, implies that $h_i(\cdot)$ and $h_{-i}(\cdot)$ are also smooth and positive on $[0, \overline{v}]$. It follows that $r_i(\cdot)$ is smooth on $[0, \overline{v}]$ (including the upper boundary $\overline{v}$).

Next, we show that $G_i^{-1}(t)$ is Hölder α-continuous with $\alpha = 1/2$. Since

$$1 - G_i(v) = (1 - F_i(v)(1 - F_{-i}(v)) = h_i(v)h_{-i}(v)(\overline{v} - v)^2,$$

it follows that $G_i'(\overline{v}) = 0$ and $G_i''(\overline{v}) = -2h_i(\overline{v})h_{-i}(\overline{v}) < 0$. Using our assumption that the densities $f_i(\cdot)$ are C^∞ on $[0, \overline{v}]$, the Morse Lemma[21] implies that there exists a diffeomorphism $q : [0, \overline{v}] \to [0, 1]$ (a smooth function with a smooth inverse) such that

$$1 - G_i(v) = q(\overline{v} - v)^2.$$

Inverting this relationship yields

$$G_i^{-1}(t) = \overline{v} - q^{-1}(\sqrt{1 - t}),$$

which implies that $G_i^{-1}(t)$ is Hölder α-continuous with $\alpha = 1/2$ as a composition of a smooth function and $\sqrt{1 - t}$. Finally, $\mu_i'(t) = r_i(G_i^{-1}(t))$ is also Hölder 1/2-continuous as a composition of a continuous $r_i(\cdot)$ and Hölder 1/2-continuous $G_i^{-1}(t)$. □

The population functions F_i, S_i, G_i, G_i^0, and μ_i as well as their estimators can be viewed as elements of the metric space $\mathbb{D}$ of *cadlag* functions equipped with the uniform norm $\|\cdot\|$. Our estimation procedure is driven by $\hat{G}_i$, $\hat{G}_i^0$, and other empirical distributions involving the bids $\{B_{il}\}$. The following lemma presents important properties of those estimators, as well as those of $\hat{\mu}_i$. Let $\rightsquigarrow$ denote the weak convergence.

Lemma B.3. *The following results hold jointly for all i's.*

(a) $\left(\sqrt{L}(\hat{G}_{i,L} - G_i), \sqrt{L}(\hat{G}_{i,L}^0 - G_i^0)\right) \rightsquigarrow \left(\mathbb{G}_i, \mathbb{G}_i^0\right)$, *where* $\mathbb{G}_i$ *and* $\mathbb{G}_i^0$ *are two correlated Gaussian processes on* $[0, \overline{v}]$.

(b) Under Assumption 1, $\sqrt{L}(\hat{\mu}_{i,L} - \mu_i) \rightsquigarrow \mathbb{M}_i$, *where for* $t \in [0, 1]$,

$$\mathbb{M}_i(t) = \mathbb{G}_i^0(G_i^{-1}(t)) - \mathbb{G}_i(G_i^{-1}(t))\mu_i'(t).$$

Furthermore, $P(\mathbb{M}_i(\cdot) \in \mathscr{H}_\alpha) = 1$ *for any* $\alpha < 1/2$.

[21] See Guillemin and Pollack (1974), p. 42.

(c) *Under Assumption 1, there exists a version of the Gaussian process* $\mathbb{M}_i$ *such that for any* $\alpha < 1/2$,

$$\limsup_{L\to\infty} L^{\alpha/2} \left\| \sqrt{L}(\hat{\mu}_{i,L} - \mu_i) - \mathbb{M}_i \right\| < \infty \quad a.s.$$

Remark 5.

1. Part (a) of Lemma B.3 is a standard Functional CLT result for Empirical Processes, see van der Vaart (1998), Theorem 19.5. In fact, the result holds jointly with the weak convergence in (30) for other empirical distributions involving the bids $\{B_{il}\}$.

2. The first claim in part (b) of the lemma follows from part (a) by the FDM, see van der Vaart (1998), Lemma 20.10 and Lemma 21.3 for quantile functions. Note that Lemma B.2 implies that μ_i' is a bounded function. The α-Hölder continuity result holds by (i) the α-Hölder continuity for of μ_i' with $\alpha = 1/2$ shown in Lemma B.2, and (ii) because the sample paths of $\mathbb{G}_i$ and $\mathbb{G}_i^0$ are α-Hölder continuous with probability one for any $\alpha < 1/2$, see for example Revuz and Yor (1999), Theorem 2.2.

3. Part (c) uses a point-wise approximation of empirical processes by Gaussian processes, see van der Vaart (1998), page 268, and Hölder continuity of μ_i' in Lemma B.2.

Proof of Lemma B.3. To simplify the notation, we omit bidder's index i in whenever there is no risk of confusion.

To show part (b), for a CDF G, let $q(G) = G^{-1}$ be the quantile transformation. By Lemma 21.3 in van der Vaart (1998), the Hadamard derivative of q (tangentially to the set of continuous functions h), is $q_G'(h) = -h(G^{-1})/g(G^{-1})$, where g is the PDF of G. We have:

$$\begin{aligned}
& \frac{1}{\delta_L}\left((G^0 + \delta_L h_L^0)(q(G + \delta_L h_L)) - G^0(q(G))\right) \\
= \;& h_L^0(q(G + \delta_L h_L)) + \frac{1}{\delta_L}\left(G^0(q(G + \delta_L h_L)) - G^0(q(G))\right) \\
\to \;& h^0(q(G)) + g^0(q(G))q_G'(h) \\
= \;& h^0(G^{-1}) - \frac{g^0(G^{-1})}{g(G^{-1})}h(G^{-1}),
\end{aligned}$$

where the convergence holds in the uniform norm for all $(h_L^0, h_L) \to (h^0, h)$ as $\delta_L \to 0$

tangentially to the set of continuous functions h. This concludes the proof of the first claim in part (b).

To show the α-Hölder continuity result in (b), write $\mathbb{M}(t+\delta) - \mathbb{M}(t) = \mathbb{G}^0(G^{-1}(t+\delta)) - \mathbb{G}^0(G^{-1}(t)) + \mathbb{G}(G^{-1}(t))(\mu'(t) - \mu'(t+\delta)) - (\mathbb{G}(G^{-1}(t+\delta)) - \mathbb{G}(G^{-1}(t)))\mu'(t+\delta)$. For any $\alpha < 1/2$,

$$|\mathbb{G}^0(G^{-1}(t+\delta)) - \mathbb{G}^0(G^{-1}(t))| \;\leq\; C_1|G^{-1}(t+\delta) - G^{-1}(t+\delta)|^\alpha \leq C_1 C_2^\alpha |\delta|^\alpha,$$

where the first inequality follows because $\mathbb{G}^0 \in \mathscr{H}_\alpha$ for any $\alpha < 1/2$ by Theorem 2.2 in Revuz and Yor (1999), and the second inequality holds because G^{-1} is continuously differentiable and, therefore, Lipschitz. By Lemma B.2,

$$|\mathbb{G}(G^{-1}(t))(\mu'(t+\delta) - \mu'(t))| \;\leq C|\delta|^{1/2} \sup\nolimits_{v\in[0,\overline{v}]} |\mathbb{G}(v)|.$$

Lastly, for any $\alpha < 1/2$,

$$|\mu'(t+\delta)(\mathbb{G}(G^{-1}(t+\delta)) - \mathbb{G}(G^{-1}(t)))| \;\leq\; C|\delta|^\alpha \sup_{t\in[0,1]} |\mu'(t)|,$$

where $\sup_{t\in[0,1]} |\mu'(t)| < \infty$ by Lemma B.2.

To show part (c), recall that both $\hat{G}_i$ and $\hat{G}_i^0$ are driven by the same random variable B_{il}. Let $\delta_L = 1/\sqrt{L}$, and $\rho_L = \delta_L(\log L)^2$. By the last result on page 268 in van der Vaart (1998), there are Gaussian processes $\mathbb{G}$ and $\mathbb{G}^0$ such that:

$$\limsup_{L\to\infty} \rho_L^{-1} \left\|\sqrt{L}(\hat{G} - G) - \mathbb{G}\right\| \;<\; \infty \quad \text{a.s.,} \tag{44}$$

$$\limsup_{L\to\infty} \rho_L^{-1} \left\|\sqrt{L}(\hat{G}^0 - G^0) - \mathbb{G}^0\right\| \;<\; \infty \quad \text{a.s..} \tag{45}$$

Define $\hat{\mathbb{G}} = \sqrt{L}(\hat{G}_L - G)$, and $\hat{\mathbb{G}}^0 = \sqrt{L}(\hat{G}_L^0 - G^0)$.

$$\begin{aligned}
\sqrt{L}(\hat{\mu} - \mu) &= \sqrt{L}(\hat{G}^0(\hat{G}^{-1}) - G^0(G^{-1})) \\
&= \frac{1}{\delta_L}\Big((G^0 + \delta_L\hat{\mathbb{G}}^0)(q(G + \delta_L\hat{\mathbb{G}}) - G^0(q(G))\Big) \\
&= \hat{\mathbb{G}}^0(q(G + \delta_L\hat{\mathbb{G}})) + g^0(q(G + \delta_L^*\hat{\mathbb{G}}))\frac{1}{\delta_L}(q(G + \delta_L\hat{\mathbb{G}}) - q(G)),
\end{aligned} \tag{46}$$

where $0 \leq \delta_L^* \leq \delta_L$ denotes a generic mean value.

For $0 < \alpha < 1/2$, pick $\epsilon_L = O(\rho_L^{1/\alpha})$. As in the proof of Lemma 21.3 in van der Vaart (1998),

$$(G + \delta_L\hat{\mathbb{G}})(q(G + \delta_L\hat{\mathbb{G}}) - \epsilon_L)) \leq \quad G(q(G)) \quad \leq (G + \delta_L\hat{\mathbb{G}})(q(G + \delta_L\hat{\mathbb{G}})).$$

Moreover,

$$\begin{aligned}
& \left\|\hat{\mathbb{G}}(q(G + \delta_L\hat{\mathbb{G}}) - \epsilon_L)) - \hat{\mathbb{G}}(q(G + \delta_L\hat{\mathbb{G}})))\right\| \\
\leq\ & 2\left\|\hat{\mathbb{G}} - \mathbb{G}\right\| + \left\|\mathbb{G}(q(G + \delta_L\hat{\mathbb{G}}) - \epsilon_L)) - \mathbb{G}(q(G + \delta_L\hat{\mathbb{G}})))\right\| \\
=\ & O_p(\rho_L) + C\left\|q(G + \delta_L\hat{\mathbb{G}}) - \epsilon_L) - q(G + \delta_L\hat{\mathbb{G}}))\right\|^\alpha \\
=\ & O_p(\rho_L),
\end{aligned}$$

where the equality in the line before the last holds by the definition of ϵ_L, (44) and α-Hölder continuity of the Gaussian process and because q is Lipschitz. Therefore,

$$\hat{\mathbb{G}}(q(G + \delta_L\hat{\mathbb{G}}))) + O_{a.s.}(\rho_L) \leq \quad \tfrac{G(q(G)) - G(q(G + \delta_L\hat{\mathbb{G}}))}{\delta_L} \quad \leq \hat{\mathbb{G}}(q(G + \delta_L\hat{\mathbb{G}})),$$

or

$$\frac{q(G + \delta_L\hat{\mathbb{G}}) - q(G)}{\delta_L} \quad = \quad -\frac{\hat{\mathbb{G}}(q(G + \delta_L\hat{\mathbb{G}})))}{g(q(G + \delta_L^*\hat{\mathbb{G}}))} + O_p(\rho_L). \tag{47}$$

Let $r(\cdot)$ be as in (43). Using (46) and (47), we obtain:

$$\begin{aligned}
\left\|\sqrt{L}(\hat{\mu} - \mu) - \mathbb{M}\right\| \quad \leq \quad & \left\|\hat{\mathbb{G}}^0(q(G + \delta_L\hat{\mathbb{G}})) - \mathbb{G}^0(q(G))\right\| \\
& + \left\|r(q(G + \delta_L^*\hat{\mathbb{G}}))\hat{\mathbb{G}}(q(G + \delta_L\hat{\mathbb{G}}))) - r(q(G))\mathbb{G}(q(G))\right\|.
\end{aligned}$$

The first term on the right-hand side can be bounded by

$$\left\|\mathbb{G}^0(q(G + \delta_L\hat{\mathbb{G}})) - \mathbb{G}^0(q(G))\right\| + O_p(\rho_L) \quad = \quad O_p(\delta_L^\alpha + \rho_L),$$

for any $\alpha < 1/2$, where we used $\|\hat{\mathbb{G}}\| \leq \|\mathbb{G}\| + O_p(\rho_L)$. The second term can be bounded by

$$\left\|r(q(G + \delta_L^*\hat{\mathbb{G}})) - r(q(G))\right\| \|\hat{\mathbb{G}}\| + \left\|\hat{\mathbb{G}}(q(G + \delta_L\hat{\mathbb{G}}))) - \mathbb{G}(q(G))\right\| \|r\|$$

$$= O_p\left(\delta_L^{1/2} + \delta_L^{\alpha}\right).$$

The result in part (c) follows from the last three displays.

□

The following lemma establishes the weak convergence of the trimmed quantile-transformed estimator $\tilde{S}_i(t) = \hat{S}_i(t \wedge t_L)$.

Lemma B.4. *For $t \in [0,1]$, let*

$$\phi(\mu_i)(t) = 1 - \exp\left(-\int_0^t \frac{d\mu_i(\tau)}{1-\tau}\right),$$
$$\phi'(h)(t) = (1 - S_i(t))\int_0^t \frac{dh(\tau)}{1-\tau},$$

where ϕ' is the functional (Hadamard) derivative of ϕ corresponding to μ_i. Define further $\phi_L(\mu_i)(t) = \phi(\mu_i)(t \wedge t_L)$, $\phi'_L(h)(t) = \phi'(h)(t \wedge t_L)$. Lastly, let

$$\mathbb{D}_0 = \{h \in \mathbb{D}[0,1] : h \in \mathscr{H}_\alpha \textit{ for any } \alpha < 1/2, h(0) = 0\}. \tag{48}$$

The following results hold jointly for all i's:

(a) For all sequences h_L such that $\|h_L - h\| = O(\delta_L^\alpha)$ for some $h \in \mathbb{D}_0$ and $0 < \alpha < 1/2$,

$$\left\|\frac{\phi_L(\mu_i + \delta_L h_L) - \phi_L(\mu_i)}{\delta_L} - \phi'_L(h_L)\right\| \to 0, \tag{49}$$

provided that as $\delta_L \to 0$ and $1 - t_L \to 0$,

$$\frac{\delta_L^{1+\alpha}}{1-t_L} = O(1), \quad \frac{\delta_L}{(1-t_L)^{1-\alpha}} = O(1). \tag{50}$$

(b) Under Assumption 1, $\|\tilde{S}_i - S_i\| \to_p 0$ and $\sqrt{L}(\tilde{S}_i - S_i) \rightsquigarrow \phi'(\mathbb{M}_i)$, provided that t_L satisfies the conditions in (50) with $\delta_L = 1/\sqrt{L}$, and $(1-t_L)\sqrt{L} \to 0$.

Remarks.

1. The modulus of continuity condition for h in the definition of $\mathbb{D}_0$ in (48) can be imposed by part (b) of Lemma B.3.

2. The result in part (a) of Lemma B.4 is Hadamard differentiability tangentially to $\mathbb{D}_0$ for trimmed functionals with a sample-dependent trimming. In this result, the linearization error is effectively controlled and negligible on the expanding interval $[0, t_L]$. Furthermore, unlike the standard tangential Hadamard differentiability, we require that the sequences h_L converge to elements of $\mathbb{D}_0$ at a sufficiently fast rate, which is justified by the strong approximation rate in Lemma B.3 (c).

3. The results in parts (b) of Lemma B.4 are the uniform consistency of the trimmed estimator of S_i for its untrimmed population counterpart, and the weak convergence of the trimmed estimator of S_i. Note that, in the weak convergence result, we use the untrimmed population object for re-centering. Similarly, the limiting process involves the untrimmed functional ϕ'. Thus, the trimming has no asymptotic effect on estimation. This is in part due to the condition $\sqrt{L}(1-t_L) \to 0$, which implies that the trimming parameter t_L must approach 1 at a rate faster than $\sqrt{L}$.

4. The conditions on the trimming parameter t_L in part (b) ensure that the approximation error in the definition of Hadamard differentiability in (49) is negligible. The rate in the first condition is determined by the approximation of the empirical process by $\mathbb{M}_i$ in Lemma B.3(c). The second rate is driven by the α-Hölder continuity of the limiting process $\mathbb{M}_i$.

5. All the conditions imposed on t_L in Lemma B.4 can be satisfied, for example, by choosing
$$1 - t_L = L^{-\beta}, \text{ with } 1/2 < \beta < 3/4.$$
as in Assumption 5. With such a choice, $(1-t_L)\sqrt{L} = L^{-\beta+1/2} \to 0$. The first condition in (50) holds as $L^{-1/2(1+\alpha)+\beta} \to 0$ or $\beta \leq (1+\alpha)/2$, since α can be chosen arbitrarily close to 1/2. The second condition in (50) implies $\beta \leq 1/(2(1-\alpha)) < 1$, where the last inequality is again due to the fact that α can be chosen arbitrarily close to 1/2. Hence, the second condition in (50) is non-binding. Thus, the rate of convergence on the trimming parameter is driven mainly by the approximation in Lemma B.3(c).

Proof of Lemma B.4. To simplify the notation, we omit bidder's index i.

For part (a), direct calculations show:

$$\frac{1}{\delta_L}\left(\phi_L(\mu + \delta_L h_L)(t) - \phi_L(\mu)(t)\right)$$

$$
\begin{aligned}
&= \exp\left(-\int_0^{t\wedge t_L} \frac{d\mu(\tau)}{1-\tau}\right)\frac{1}{\delta_L}\left(1-\exp\left(-\delta_L\int_0^{t\wedge t_L}\frac{dh_L(\tau)}{1-\tau}\right)\right)\\
&= (1-S(t\wedge t_L))\int_0^{t\wedge t_L}\frac{dh_L(\tau)}{1-\tau}\\
&\quad +0.5\,(1-S(t\wedge t_L))\,\delta_L\left(\int_0^{t\wedge t_L}\frac{dh_L(\tau)}{1-\tau}\right)^2\exp\left(-\delta_L^*\int_0^{t\wedge t_L}\frac{dh_L(\tau)}{1-\tau}\right), \qquad (51)
\end{aligned}
$$

where the second equality follows by the mean-value expansion of $1-\exp(-sx)$ around $s=0$, and δ_L^* is the mean-value: $0\le\delta_L^*\le\delta_L$.

Using integration by parts,

$$
\begin{aligned}
\int_0^{t\wedge t_L}\frac{dh_L(\tau)}{1-\tau} &= \frac{h_L(t\wedge t_L)}{1-t\wedge t_L}-\int_0^{t\wedge t_L}h_L(\tau)d\left(\frac{1}{1-\tau}\right)\\
&= \frac{h(t\wedge t_L)}{1-t\wedge t_L}-\int_0^{t\wedge t_L}h(\tau)d\left(\frac{1}{1-\tau}\right)+O\left(\frac{\delta_L^\alpha}{1-t\wedge t_L}\right)\\
&= \int_0^{t\wedge t_L}\frac{dh(\tau)}{1-\tau}+O\left(\frac{\delta_L^\alpha}{1-t\wedge t_L}\right), \qquad (52)
\end{aligned}
$$

where the big-O term is uniform in t and we used the condition $\|h_L-h\|=O(\delta_L^\alpha)$. Moreover, since $h\in\mathscr{H}_\alpha$ for any $\alpha<1/2$ and $h(1)=0$,

$$
\begin{aligned}
\int_0^{t\wedge t_L}\frac{dh(\tau)}{1-\tau} &= -\frac{h(1)-h(t\wedge t_L)}{1-t\wedge t_L}+\int_0^{t\wedge t_L}(1-\tau)^\alpha\frac{h(1)-h(\tau)}{(1-\tau)^\alpha}d\left(\frac{1}{1-\tau}\right)\\
&\quad +h(1)\left(\frac{1}{1-t\wedge t_L}-\int_0^{t\wedge t_L}d\left(\frac{1}{1-\tau}\right)\right)\\
&= O\left(\frac{1}{(1-t\wedge t_L)^{1-\alpha}}\right)+O(1)\int_0^{t\wedge t_L}(1-\tau)^{\alpha-2}d\tau+h(1)\\
&= O\left(1+\frac{1}{(1-t\wedge t_L)^{1-\alpha}}\right), \qquad (53)
\end{aligned}
$$

where the $O(1)$ terms are uniform in t. Also, since S is differentiable,

$$
\sup_{t\in[0,1]}\left|\frac{1-S(t\wedge t_L)}{1-t\wedge t_L}\right|=O(1). \qquad (54)
$$

By (52), (53), and (54),

$$
(1-S(t\wedge t_L))\delta_L\left(\int_0^{t\wedge t_L}\frac{dh_L(\tau)}{1-\tau}\right)^2 = \delta_L O(1-t\wedge t_L)O\left(1+\frac{\delta_L^\alpha}{1-t\wedge t_L}+\frac{1}{(1-t\wedge t_L)^{1-\alpha}}\right)^2
$$

$$= O\left(\frac{\delta_L^{1/2+\alpha}}{(1-t\wedge t_L)^{1/2}} + \frac{\delta_L^{1/2}}{(1-t\wedge t_L)^{1/2-\alpha}}\right)^2,$$

and, since $1 - t_L \to 0$,

$$\sup_{t\in[0,1]}\left|(1-S(t\wedge t_L))\delta_L\left(\int_0^{t\wedge t_L}\frac{dh_L(\tau)}{1-\tau}\right)^2\right| = O\left(\frac{\delta_L^{1/2+\alpha}}{(1-t_L)^{1/2}} + \frac{\delta_L^{1/2}}{(1-t_L)^{1/2-\alpha}}\right)^2 \quad (55)$$

where the first term in the O-expression is due to approximation of the empirical process by a Gaussian process, and the second term is due to the α-Hölder continuity of the limiting process. Next, consider the exponential term in (51). By (52) and (53),

$$\sup_{t\in[0,1]}\left|\delta_L\int_0^{t\wedge t_L}\frac{dh_L(\tau)}{1-\tau}\right| = O\left(\delta_L\left(1+\frac{1}{(1-t_L)^{1-\alpha}}\right) + \frac{\delta_L^{1+\alpha}}{1-t_L}\right). \quad (56)$$

Here, the first term in the O-expression is due to α-Hölder continuity of the limiting process, and the second term is due to the approximation of h_L by a Gaussian process. Lastly, by (51), (55), and (56), for h_L's such that $\|h_L - h\| = O(\delta_L^\alpha)$ and $h \in \mathbb{D}_0$,

$$\begin{aligned}
&\left\|\frac{1}{\delta_L}\Big(\phi_L(\mu+\delta_L h_L)(t) - \phi_L(\mu)(t)\Big) - (1-S(t\wedge t_L))\int_0^{t\wedge t_L}\frac{dh_L(\tau)}{1-\tau}\right\| \\
&= O\left(\frac{\delta_L^{1/2+\alpha}}{(1-t_L)^{1/2}} + \frac{\delta_L^{1/2}}{(1-t_L)^{1/2-\alpha}}\right)^2 \exp\left(O\left(\delta_L + \frac{\delta_L}{(1-t_L)^{1-\alpha}} + \frac{\delta_L^{1+\alpha}}{1-t_L}\right)\right) \\
&= o(1)\exp\left(O(1)\right),
\end{aligned}$$

where the last equality holds by (50).

To show the uniform consistency in part (b), in place of h_L we use $\hat{\mathbb{M}} = \sqrt{L}(\hat{\mu} - \mu)$, which satisfies the conditions imposed on h_L in part (a) of the lemma.

$$\begin{aligned}
\left\|\tilde{S}_L - S\right\| &= \left\|\phi_L(\mu + L^{-1/2}\hat{\mathbb{M}}) - \phi(\mu)\right\| \\
&\leq \left\|\phi_L(\mu + L^{-1/2}\hat{\mathbb{M}}) - \phi_L(\mu)\right\| + \|\phi_L(\mu) - \phi(\mu)\| \\
&\leq L^{-1/2}\left\|\phi_L'(\hat{\mathbb{M}})\right\| + \sup_{t\in[t_L,1]}(S(t) - S(t_L)), \quad (57)
\end{aligned}$$

where the inequality in the last line holds by part (a) of the lemma (for the first term) and because $\phi_L(t) = \phi(t)$ for $t \leq t_L$ (for the second term). Since S is differentiable with a bounded derivative, and because for $t \geq t_L$ we have $t - t_L \leq 1 - t_L$, the second term in (57) is of order

$$\sup_{t\in[t_L,1]} (S(t) - S(t_L)) = O(1 - t_L) = o(1). \tag{58}$$

Moreover, for h_L that satisfies the conditions from part (a) of the lemma, by (52) and (54) we have

$$\sup_{t\in[0,1]} \left| (1 - S(t \wedge t_L)) \left(\int_0^{t\wedge t_L} \frac{dh_L(\tau)}{1-\tau} - \int_0^{t\wedge t_L} \frac{dh(\tau)}{1-\tau} \right) \right| = O\left(\delta_L^{\alpha}\right). \tag{59}$$

It follows from (53), (54), and (59) that $\phi_L'(\hat{\mathbb{M}})(t)$ in (57) of order

$$\delta_L O(1 - t \wedge t_L) O_p \left(\frac{1}{1 - t \wedge t_L} \right)^{1-\alpha} = o_p(1)$$

uniformly in t, which concludes the proof of the uniform consistency in part (b).

To show the weak convergence result in part (b), we verify the conditions of Lemma A.1 with $r_L = 1/\delta_L = \sqrt{L}$. For condition (i), as in (57) and (58), $\sqrt{L}\|\phi_L(\mu) - \phi(\mu)\| = O(\sqrt{L}(1 - t_L)) = o(1)$, where the second equality is by the conditions imposed on t_L in part (b) of Lemma B.4. Condition (ii) of Lemma A.1 has been established in part (a) of Lemma B.4. Condition (iv) holds by Lemma B.3(b).

To show that condition (iii) of Lemma A.1 holds, first note that $\|\phi_L'(h_L) - \phi_L'(h)\| \to 0$ for $\|h_L - h\| = O(\delta_L^{\alpha})$, where the latter condition is satisfied by $\hat{\mathbb{M}}$ with probability approaching one due to Lemma B.3(c) with $\delta_L = 1/\sqrt{L}$:

$$\begin{aligned} \|\phi_L'(h_L) - \phi_L'(h)\| &= \sup_{t\in[0,t_L]} \left| (1 - S(t \wedge t_L)) \int_0^{t\wedge t_L} \frac{d(h_L(\tau) - h(\tau))}{1-\tau} \right| \\ &= O(\delta_L^{\alpha}), \end{aligned}$$

where the equality in the second line holds by (54). Next, $\phi_L'(h)(t) - \phi'(h)(t) = 0$ for $t \leq t_L$. For $t \geq t_L$,

$$\begin{aligned} \phi_L'(h)(t) - \phi'(h)(t) &= (1 - S(t_L)) \int_0^{t_L} \frac{dh(\tau)}{1-\tau} - (1 - S(t)) \int_0^{t} \frac{dh(\tau)}{1-\tau} \\ &= (S(t) - S(t_L)) \int_0^{t_L} \frac{dh(\tau)}{1-\tau} - (1 - S(t)) \int_{t_L}^{t} \frac{dh(\tau)}{1-\tau} \end{aligned}$$

$$= O\left(1-t_L\right)^{\alpha} - (1-S(t))\int_{t_L}^{t}\frac{dh(\tau)}{1-\tau},$$

where the equality in the last line holds by (53) and (58), and the big-O term is uniform in t. For the second term in the last display, consider

$$\sup_{t\in[t_L,1]}\left|(1-S(t))\int_{t_L}^{t}\frac{dh(\tau)}{1-\tau}\right| = \left|(1-S(t_L^*))\int_{t_L}^{t_L^*}\frac{dh(\tau)}{1-\tau}\right|$$

for some t_L^* such that $t_L \le t_L^* \le 1$. If $t_L^* < 1$,

$$\begin{aligned}\left|(1-S(t_L^*))\int_{t_L}^{t_L^*}\frac{dh(\tau)}{1-\tau}\right| &\le \left|(1-S(t_L^*))\int_{0}^{t_L^*}\frac{dh(\tau)}{1-\tau}\right| + \left|(1-S(t_L^*))\int_{0}^{t_L}\frac{dh(\tau)}{1-\tau}\right| \quad (60)\\ &= O\left(1-t_L^*\right)^{\alpha} + O\left(1-t_L\right)^{\alpha}. \quad (61)\end{aligned}$$

If $t_L^* = 1$, take the limit of the expression in (60) as $t_L^* \to 1$ to obtain convergence to zero due to (61), which concludes the proof of part (b). □

We can now state the proof of Proposition 4.

Proof of Proposition 4. Again, to simplify the notation, we omit bidder's index i.

Write $F(v) = \varphi(S,G)(v) \equiv S(G(v))$. The functional φ is Hadamard differentiable, and its Hadamard derivative is equal to

$$\varphi'_{S,G}(h_S,h_G)(v) = h_S(G(v)) + S'(G(v))h_G(v),$$

where $S'(t)$ denotes the derivative (density) of S at t. Therefore,

$$\begin{aligned}\sqrt{L}(\tilde{F}-F)(\cdot) &= \sqrt{L}(\varphi(\tilde{S},\hat{G}) - \varphi(S,G))\\ &\rightsquigarrow \phi'(\mathbb{M})(G(\cdot)) - S'(G(\cdot))\mathbb{G}(\cdot)\\ &= \phi'(\mathbb{M})(G(\cdot)) - \frac{f(\cdot)\mathbb{G}(\cdot)}{g(\cdot)}, \quad (62)\end{aligned}$$

where the result in the second line holds by Lemma B.4(b) and Lemma B.3(a). The result in the last line holds since $S(t) = F(G^{-1}(t))$ and therefore $S'(G(v)) = f(v)/g(v)$ Next,

$$\phi'(\mathbb{M})(G(v)) = (1-S(G(v)))\int_0^{G(v)}\frac{d\mathbb{M}(\tau)}{1-\tau}$$

$$= \quad (1 - F(v)) \int_0^v \frac{d\mathbb{M}(G(u))}{1 - G(u)}, \tag{63}$$

where the equality in the second line holds by a change of variable $u = G^{-1}(\tau)$. By the definition of $\mathbb{M}_i$ in Lemma B.3(b),

$$\begin{aligned}
\int_0^v \frac{d\mathbb{M}(G(u))}{1 - G(u)} &= \int_0^v \frac{d\mathbb{G}^0(u)}{1 - G(u)} - \int_0^v \frac{d\left(\mathbb{G}(u)\mu'(G(u))\right)}{1 - G(u)} \\
&= \int_0^v \frac{d\mathbb{G}^0(u)}{1 - G(u)} - \frac{\mathbb{G}(v)\mu'(G(v))}{1 - G(v)} + \int_0^v \frac{\mathbb{G}(u)\mu'(G(u))dG(u)}{(1 - G(u))^2}, \quad (64)
\end{aligned}$$

where the equality in the second line holds by integration by parts. Since $\mu(t) = G^0(G(u))$, $\mu'(G(u)) = g^0(u)/g(u)$ and therefore,

$$\mu'(G(u))dG(u) = dG^0(u). \tag{65}$$

Lastly, by our basic decensoring formula (2),

$$\frac{\mu'(G(v))}{1 - G(v)} = \frac{g^0(v)}{g(v)(1 - G(v))} = \frac{f(v)}{(1 - F(v))g(v)}. \tag{66}$$

The result of the proposition now follows from (62)–(66). □

Proof of Proposition 6. We omit bidder's index i when there is no risk of confusion.

We show (35) first. Following the definition of μ in (41), we define

$$\hat{\mu}^\dagger(t) = \hat{G}^{0,\dagger}((\hat{G}^\dagger)^{-1}(t)).$$

Following the definition of S in (39) and (42), we also define

$$\hat{S}^\dagger(t) = \hat{F}^\dagger((\hat{G}^\dagger)^{-1}(t)) = 1 - \exp\left(-\int_0^t \frac{d\hat{\mu}^\dagger(\tau)}{1 - \tau}\right),$$

and a trimmed bootstrap estimator

$$\tilde{S}^\dagger(t) = \hat{S}^\dagger(t \wedge t_L) = 1 - \exp\left(-\int_0^{t \wedge t_L} \frac{d\hat{\mu}^\dagger(\tau)}{1 - \tau}\right).$$

By adapting the proof of Lemma 21.3 in van der Vaart (1998) and as in the proof of

Lemma B.3(b), we can write

$$\begin{aligned}\sqrt{L}(\hat{\mu}-\mu) &= \sqrt{L}\Big(\hat{G}^0(G^{-1})-G^0(G^{-1})\Big)-\frac{g^0(G^{-1})}{g(G^{-1})}\sqrt{L}\Big(\hat{G}(G^{-1})-\tau\Big)\\ &\quad +o_p\Big(\sqrt{L}\Big(\hat{G}^0(G^{-1})-G^0(G^{-1})\Big)+\sqrt{L}\Big(\hat{G}(G^{-1})-\tau\Big)\Big), \qquad (67)\\ \sqrt{L}(\hat{\mu}^\dagger-\mu) &= \sqrt{L}\Big(\hat{G}^{0,\dagger}(G^{-1})-G^0(G^{-1})\Big)-\frac{g^0(G^{-1})}{g(G^{-1})}\sqrt{L}\Big(\hat{G}^\dagger(G^{-1})-\tau\Big)\\ &\quad +o_p\Big(\sqrt{L}\Big(\hat{G}^{0,\dagger}(G^{-1})-G^0(G^{-1})\Big)+\sqrt{L}\Big(\hat{G}^\dagger(G^{-1})-\tau\Big)\Big),\end{aligned}$$

where the o_p term is uniform in τ, and therefore,

$$\begin{aligned}\sqrt{L}(\hat{\mu}^\dagger-\hat{\mu}) &= \sqrt{L}\Big(\hat{G}^{0,\dagger}(G^{-1})-\hat{G}^0(G^{-1})\Big)-\frac{g^0(G^{-1})}{g(G^{-1})}\sqrt{L}\Big(\hat{G}^\dagger(G^{-1})-\hat{G}(G^{-1})\Big)\\ &\quad +o_p\Big(\sqrt{L}\Big(\hat{G}^{0,\dagger}(G^{-1})-\hat{G}^0(G^{-1})\Big)+\sqrt{L}\Big(\hat{G}^\dagger(G^{-1})-\hat{G}(G^{-1})\Big)\Big). \qquad (68)\end{aligned}$$

Let $\tilde{G}$ and $\tilde{G}^0$ denote estimators constructed using independent copies of the original data. By Proposition 3.1 in Chen and Lo (1997),

$$\begin{aligned}\|\hat{G}^\dagger-\hat{G}-\tilde{G}+G\| &= O_{a.s.}(L^{-3/4}(\log L)^{3/4}),\\ \|\hat{G}^{0,\dagger}-\hat{G}^0-\tilde{G}^0+G^0\| &= O_{a.s.}(L^{-3/4}(\log L)^{5/4}).\end{aligned}$$

Let $\tilde{\mu}=\tilde{G}^0(\tilde{G}^{-1})$, and note that $\tilde{\mu}$ is an independent copy of $\hat{\mu}$. By taking the difference between (68) and the same expansion as in (67) applied to $\tilde{\mu}$, and applying the result of Chen and Lo (1997), we obtain that

$$\sqrt{L}\|\hat{\mu}^\dagger-\hat{\mu}-\tilde{\mu}+\mu\| = O_{a.s.}(L^{-1/4}(\log L)^{5/4}). \qquad (69)$$

Let $h_L^\dagger=\sqrt{L}(\hat{\mu}^\dagger-\hat{\mu})$. As in equation (51) in the proof of Lemma B.4(a),

$$\begin{aligned}&\sqrt{L}(\tilde{S}^\dagger(t)-\tilde{S}(t))\\ &= (1-\tilde{S}(t))\int_0^{t\wedge t_L}\frac{dh_L^\dagger(\tau)}{1-\tau}\\ &\quad +0.5\left(1-\tilde{S}(t)\right)\delta_L\left(\int_0^{t\wedge t_L}\frac{dh_L^\dagger(\tau)}{1-\tau}\right)^2\exp\left(-\delta_L^*\int_0^{t\wedge t_L}\frac{dh_L^\dagger(\tau)}{1-\tau}\right). \qquad (70)\end{aligned}$$

Next, let $\tilde{h}_L = \sqrt{L}(\tilde{\mu} - \mu)$ and $\epsilon_L = h_L^\dagger - \tilde{h}_L$.We have:

$$\begin{aligned}\int_0^{t\wedge t_L} \frac{dh_L^\dagger(\tau)}{1-\tau} &= \frac{h_L^\dagger(t\wedge t_L)}{1-t\wedge t_L} - \int_0^{t\wedge t_L} \frac{h_L^\dagger(\tau)d\tau}{(1-\tau)^2} \\ &= \int_0^{t\wedge t_L} \frac{d\tilde{h}_L(\tau)}{1-\tau} + \frac{\epsilon_L(t\wedge t_L)}{1-t\wedge t_L} - \int_0^{t\wedge t_L} \frac{\epsilon_L(\tau)d\tau}{(1-\tau)^2} \\ &= \int_0^{t\wedge t_L} \frac{d\tilde{h}_L(\tau)}{1-\tau} + O_{a.s.}\left(\frac{(\log L)^{5/4}}{L^{1/4}(1-t\wedge t_L)}\right), \qquad (71)\end{aligned}$$

where the equality in the last line is due to the definition of ϵ_L and by (69), and the $O_{a.s.}$ term is uniform in t.

Since $\sqrt{L}(\tilde{S}-S) \rightsquigarrow \phi'(\mathbb{M})$ by Lemma B.4(b), ϕ' is linear, $\mathbb{M}$ is Gaussian and α-Hölder-continuous for $\alpha < 1/2$, and $\mathbb{M}(1) = 0$, it follows that $\sqrt{L}(\tilde{S}(t) - S(t))/(1-t\wedge t_L)^\alpha = O_p(1)$ uniformly in t for $\alpha < 1/2$, and

$$\begin{aligned}1 - \tilde{S}(t) &= (1 - S(t\wedge t_L))\left(1 - \frac{\sqrt{L}(\tilde{S}(t) - S(t\wedge t_L))}{\sqrt{L}(1 - S(t\wedge t_L))}\right) \\ &= (1 - S(t\wedge t_L))\left(1 + O_p\left(\frac{1}{\sqrt{L}(1-t\wedge t_L)^{1-\alpha}}\right)\right) \\ &= (1 - S(t\wedge t_L))(1 + o_p(1)).\end{aligned}$$

The equality in the last line holds by $1 - t_L = L^{-\beta}$ with $\beta < 3/4$ and since α can be chosen arbitrarily close to $1/2$; moreover the o_p term is uniform in t. Hence, by (71),

$$\begin{aligned}(1 - \tilde{S}(\cdot)) \int_0^{\cdot\wedge t_L} \frac{dh_L^\dagger(\tau)}{1-\tau} &= (1 + o_p(1))(1 - S(\cdot\wedge t_L)) \int_0^{\cdot\wedge t_L} \frac{d\tilde{h}_L(\tau)}{1-\tau} + O_p\left(L^{-1/4}(\log L)^{5/4}\right) \\ &\rightsquigarrow \phi'(\mathbb{M}^\dagger(\cdot)), \qquad (72)\end{aligned}$$

where $\mathbb{M}^\dagger$ is an independent copy of $\mathbb{M}$ since $\tilde{\mu}$ is an independent copy of $\hat{\mu}$.

Similarly to (55) in the proof of Lemma B.4(b), since $\delta_L = 1/\sqrt{L}$, and by (71),

$$\begin{aligned}&\sup_{t\in[0,1]} \left| (1 - S(t\wedge t_L))\delta_L \left(\int_0^{t\wedge t_L} \frac{dh_L^\dagger(\tau)}{1-\tau}\right)^2 \right| \\ &= O_p\left(\frac{\delta_L^{1/2+\alpha}}{(1-t_L)^{1/2}} + \frac{\delta_L^{1/2}}{(1-t_L)^{1/2-\alpha}} + \frac{(\log L)^{5/4}}{L^{1/2}(1-t_L)^{1/2}}\right)^2\end{aligned}$$

$$= \quad o_p(1). \tag{73}$$

Similarly to (56) in the proof of Lemma B.4(b) and by (71),

$$\begin{aligned}\sup_{t\in[0,1]}\left|\delta_L\int_0^{t\wedge t_L}\frac{dh_L^\dagger(\tau)}{1-\tau}\right| &= O_p\left(\frac{\delta_L}{(1-t_L)^{1-\alpha}}+\frac{\delta_L^{1+\alpha}}{1-t_L}+\frac{(\log L)^{5/4}}{L^{3/4}(1-t_L)}\right)\\ &= o_p(1), \end{aligned}\tag{74}$$

where the equality in the last line holds since $1-t_L = L^{-\beta}$ with $\beta < 3/4$.

By (70), (72), (73), and (74) we have that

$$\sqrt{L}(\tilde{S}^\dagger(t)-\tilde{S}(t)) \rightsquigarrow \phi'(\mathbb{M}^\dagger(\cdot)).$$

The result in (35) now follows by the FDM for the bootstrap (van der Vaart, 1998, Theorem 23.5) and the same arguments as in the proof of Proposition 4, since $\tilde{F}^\dagger = \tilde{S}^\dagger(\hat{G}^\dagger)$.

The result in (36) holds by the bootstrap FDM, Proposition 3.1 in Chen and Lo, (29), and since the functional ψ_{col} is Hadamard differentiable on $[\underline{v}_0,\overline{v}]\subset(0,\overline{v}]$.

To show (37), write

$$\sqrt{L}(\hat{\Delta}_i^\dagger(b)-\hat{\Delta}_i(b)) = \sqrt{L}(\hat{G}_i^\dagger(b)-\hat{G}_i(b)) - \sqrt{L}(\hat{G}_i^{pred,\dagger}(b)-\hat{G}_i^{pred}(b)).$$

The result in (37) follows by the bootstrap FDM and the previous results of the proposition as the functional $\psi_{i,pred}$ defined in (32) is Hadamard differentiable. □