Price Discrimination and Efficient Matching

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ABSTRACT: This paper considers the problem of a monopoly matchmaker that uses a schedule of entrance fees to sort different types of agents on the two sides of a matching market into exclusive meeting places, where agents randomly form pairwise matches. We make the standard assumption that the match value function exhibits complementarities, so that matching types at equal percentiles maximizes total match value and is efficient. We provide necessary and sufficient conditions for the revenue-maximizing sorting to be efficient. These conditions require the match value function, modified to incorporate the incentive cost of eliciting private type information, to exhibit complementarities along the efficient path of matching types at equal percentiles.

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1. Introduction

Many users of Internet dating agencies such as Match.com complain about the problem of misrepresentations and exaggerations by some users in the information they provide to the agencies.\(^1\) This problem, and the perception of it among the public, is responsible for reducing the quality of Internet search and matching and for preventing many lonely people from fully utilizing the online dating services, in spite of the advantages in cost, safety, anonymity and breadth of the reach offered by the new technology compared to more traditional means of finding dates. Although Internet dating agencies rely on individual users to report information about themselves truthfully and have little resource or capability of directly validating the information, economic theory suggests price discrimination as a way of making the reported information credible and improving match quality.\(^2\) After all, self-selection is evident in more traditional meeting places. Night clubs that cater people with more expensive tastes have higher cover charges. More exclusive singles clubs charge more for membership fees.

In this paper we look at the theoretical problem of a monopoly matchmaker that uses a schedule of entrance fees to sort different types of agents on the two sides of a matching market into different “meeting places,” in which agents are randomly pairwise matched. This problem is presented in section 2. The monopoly matchmaker faces two constraints in revenue maximization. First, the matchmaker does not observe the one-dimensional characteristic (“type”) of each agent. This information constraint means that the matchmaker must provide incentives in terms of match quality and fees for agents to self-select into the meeting places. We refer to the menu of meeting places created by the matchmaker as the “sorting structure.” Second, the monopoly matchmaker faces a technology constraint that restricts match formation in each meeting place to random


\(^2\) Most of the Internet dating agencies in North America, including the dominant companies such as Match.com, at present charge a uniform fee for all participants. Lavalife.com, an industry pioneer founded in Canada in 1987, now offers a fee schedule based on the number of initial contact messages that a participant wishes to purchase. This is not the same kind of price discrimination discussed in the present paper, and is unlikely to solve the problem of misrepresentation or improve matching efficiency.
pairwise matching. This primitive matching technology allows us to focus on the impact of revenue-maximization on the sorting structure and matching efficiency. We make the standard assumption that the match value function exhibits complementarities between types. Under this assumption, the “perfect sorting,” or matching types at equal percentiles with a continuum of meeting places, maximizes the total match value and is efficient. The goal of this paper is to understand when the perfect sorting is revenue-maximizing.

Our framework fits various two-sided market environments characterized by sorting or self-selection based on prices. For example, online job search has become a major way to explore potential employer-employee relationships. However, existing job search services such as Monster.com are plagued by job intermediaries (head hunters) that post entries only to collect information from job applicants and positions and then profit from the information. The job market and dating market share a few common features that allow our framework to apply: match characteristics of market participants can be summarized in a one dimensional type; participants on one side of the market share the same preference ordering over matches with the agents on the opposite side; and types are complementary in the match value function. Other two-sided matching markets where price-based intermediation can potentially play an important role include matching tenants to apartments, and matching loan applicants to bank loans. The results in the present paper show that a monopoly matchmaker can have the same incentive as a social planner to implement the efficient matching. In this case, the matchmaker makes directed search possible by creating one meeting place for each type and achieves the first best matching outcome, in spite of the technological constraint of random pairwise matching.

In section 3 we show how the matchmaker’s problem of designing fee schedules and the corresponding sorting structure can be transformed into a problem of monopoly price discrimination. The assumption of complementarity in the match value function implies that the standard single-crossing condition in the price discrimination literature is satisfied.

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3 A limitation of this paper is the assumption of a monopoly matchmaker, as competition exists in most two-sided markets. We believe that understanding the monopoly revenue-maximization problem is necessary for an analysis of price competition in two-sided markets where participants have heterogeneous qualities and sorting is important. See our companion paper Damiano and Li (2004) for an application of the present framework to issues of price competition.
for both sides of the market, and results in the incentive compatibility constraint that a higher type receives a higher match quality. The transformation is then achieved by combining this incentive compatibility constraint with the feasibility constraint that match qualities are generated in a two-sided matching environment where agents participate in at most one meeting place and are pairwise randomly matched in each meeting place. The outcome of the transformation is a sorting structure in which the efficient matching path in the type space (pairwise matching of types at equal percentiles) is partitioned into pooling intervals and sorting intervals: each pooling interval on the efficient matching path represents a meeting place with the corresponding intervals of types on the two sides being pooled together, while each sorting interval represents a continuum of meeting places with the types efficiently matched. We refer to this sorting structure as “weak sorting.” In weak sorting, if two types on the same side of the market participate in two different meeting places, then the higher type not only has a higher average match type, but never gets a lower match. Weak sorting allows us to rewrite the objective function of the monopolist by using a “virtual match value function,” which is the match value function adjusted for the incentive costs of eliciting private type information.

Unlike a standard price discrimination problem, the solution to the sorting structure design problem cannot be characterized by pointwise maximization due to the feasibility constraint on how the monopolist provides match quality to the two sides of the market. In section 4 we provide a necessary condition for the optimal sorting structure to be the perfect sorting. This condition requires that the virtual match value function be locally supermodular along the efficient matching path, that is, have positive cross partial derivatives at equal percentiles. If at any percentile this condition is not satisfied, the monopoly matchmaker can increase revenue by pooling adjacent types into a single meeting place. Local supermodularity of the virtual match value function along the efficient matching path is not generally sufficient for the optimal sorting structure to be the perfect sorting, because it does not guarantee that a greater revenue cannot be generated by pooling a large set of types on the two sides. A sufficient condition for the perfect sorting to be optimal is that the virtual match value function is “globally” supermodular along the efficient matching path. Intuitively, the inability to observe the type of agents creates an incentive cost for the matchmaker to extract surplus because the matchmaker has to rely on
self-selection by the agents. The perfect sorting structure maximizes revenue for the monopolist matchmaker if this incentive cost does not dominate the complementarities in the match value function. In this case, there is no distortion in match quality provision for any type, in contrast to the standard result in the price discrimination literature that quality is under-provided for all types except the highest. Finally, if the virtual match value function satisfies the stronger condition that it is supermodular over the entire type space (as opposed to global supermodularity along the efficient matching path), then the matchmaker’s revenue is increasing in the number of meeting places created. In this case, even when there are technological constraints on creation of meeting places, revenue-maximization always leads to improvement in matching efficiency.

1.1. Relation to the existing literature

A classical reference in the price discrimination literature is Maskin and Riley (1984) (see also Mussa and Rosen, 1978). In both the standard price discrimination and our sorting structure design problems, the monopolist faces consumers with one-dimensional private information about their willingness to pay, and must provide incentives for self-selection. In a price discrimination problem, the monopolist controls the quality (or quantity) of the good provided. Consumers of different types self-select by choosing a price-quality combination from the schedule offered by the monopolist. In contrast, in our sorting structure design problem the monopolist chooses a partition of the market into meeting places in which agents randomly match, and the associated fee schedules. Besides the standard incentive compatibility and participation constraints, the monopolist also faces additional feasibility constraints because the pair of quality schedules must be consistent with the sorting structure.

The most closely related paper in the price discrimination literature is Rayo (2002). He considers the price-discrimination problem of a monopolist that sells a status good. In his benchmark model, there is no intrinsic quality dimension to different varieties of the good, and buyers of one variety care only about who else are buying the same variety. Our

\footnote{We thank Jonathan Levin for alerting us to the paper.}
result of weak sorting implies that this is essentially the same price discrimination problem considered here if one restricts to a symmetric matching environment. His results on when providing different varieties to different types is optimal can therefore be obtained as a special case of our necessary and sufficient conditions for the perfect sorting to be optimal.

Inderst (2001) questions the classical result in the price discrimination literature that it is optimal for the monopolist to offer low types distorted contracts in order to extract more rents from higher types. His paper looks at contract design in a matching market environment with frictions and shows that the distortion result does not hold anymore. In particular, for low enough search frictions all contracts are free of distortion. The driving force of the result is that in a search and matching environment reservation values are type dependent as higher types will generally have more match opportunity and therefore higher reservation values. In contrast, our no-distortion result does not rely on type-dependent reservation values, and is generated by feasibility restrictions on match quality provision in a two-sided matching market.

Our paper is the first to investigate intermediation in two-sided markets with heterogeneous types and search frictions from the mechanism design point of view. In the existing literature on two-sided search, sorting of heterogeneous types occurs in equilibrium either because finding a good match takes time (Burdett and Coles, 1997; Smith, 2002), or because meeting a potential partner is costly (Morgan, 1995). Unlike these models, our framework is static and we obtain sorting as a result of maximizing revenue by an intermediary. Building on the two-sided search literature, Bloch and Ryder (2000) analyze the problem of a monopolistic matchmaker that competes with a decentralized matching market with frictions. Unlike our paper, the matchmaker observes the types and can implement perfect sorting in exchange for a fee. Due to its information advantage, the only decision for the matchmaker is what types to service given that their reservation utilities are endogenously determined in the decentralized market.

The present paper grew out of our previous work on dynamic sorting (Damiano, Li and Suen, forthcoming). The two papers share the same interest in efficiency of matching markets in the presence of search frictions. In both papers, search frictions are modeled by the primitive search technology of random meeting. In Damiano, Li and Suen,
dynamic sorting provides higher types more search opportunities and improves matching efficiency. In the present paper, price discrimination by the monopolist creates directed search markets and can achieve the efficient matching. In a companion paper (Damiano and Li, 2004), we use a simplified framework of the present paper to study how competition among matchmakers can affect the sorting structure and matching efficiency.

2. The Model

Consider a two-sided matching market. Without loss of generality, we assume that the two sides have the same size. For convenience, agents of the two sides are called men and women, respectively. Men and women have heterogeneous one-dimensional characteristics, called types. The type distribution is \( F(\cdot) \) for men and \( G(\cdot) \) for women. Both type distributions are assumed to have differentiable densities, denoted as \( f \) and \( g \), respectively. The support is \([a_m, b_m]\) for men and \([a_w, b_w]\) for women, with both subsets of \( \mathbb{R}_+ \), and \( b_m \) and \( b_w \) possibly infinite. A match between a type \( x \) man and a type \( y \) woman produces value \( xy \) to both the man and the woman, so \( 2xy \) is the total match value for the pair.\(^5\)

We assume that all market participants are risk neutral and have quasi-linear preferences. They care only about the difference between the expected match value and the entrance fee they pay. An unmatched agent gets a payoff of 0, regardless of type. Section 5 discusses how our results can be extended when reservation utilities either differ for the two sides or are type-dependent.

An important assumption about the matching preferences that we have made above is that matching characteristics of each agent can be summarized in one-dimensional type. This simplification relative to the reality of matching markets, facilitates comparison with the existing literature, where the assumption of one-dimensional type is standard. Implicit in our specification of the matching preferences is that all agents on each side of the market have homogeneous preferences. For the same price, they all prefer the highest type agents on the other side. Clearly there are matching characteristics that are ranked differently.

\(^5\) Given our later assumption of 0 payoff for unmatched agents, the payoffs are unchanged if matched couples bargain over the division of the total match value \( 2xy \) using the Nash bargaining solution.
by agents in real matching markets. For example, in online dating, it is sometimes argued that not everyone wishes to date the smartest person. Rather, matching preferences may be single peaked. However, when the most desirable match differs across agents, the competition among agents is reduced and so are the incentives to misrepresent this kind of matching characteristics. Since the present paper is about how the monopoly matchmaker uses price discrimination to mitigate the problem of misrepresentation in a matching market, we will focus on matching characteristics that all agents rank identically and compete for.\footnote{Users of online dating tend to segregate into services that cater groups that share the same preferences for non-competing characteristics. One such example is religious affiliation. Jdate.com attracts only Jewish users while Eharmony.com targets the Christian population.}

Another important assumption about the matching preferences we have made is that types are complementary in generating match values. This assumption is embedded in the match value function $xy$: each agent’s willingness to pay for an improvement in match type increases with the type of the agent.\footnote{In online dating, a more attractive individual is more likely to have a successful first date than a less attractive individual, so even if both derive the same utility from a given potential match, the more attractive individual is willing to pay more for an improvement in the quality of the potential match.} Complementarity is a standard assumption in the literature on matching. Under this assumption, matching types at equal percentiles maximizes the total value of pairwise two-sided matches and is efficient (Becker, 1981) Formally, for each $x \in [a_m, b_m]$, let

$$s_m(x) = G^{-1}(F(x))$$

be the female type at the same percentile of the male type $x$. We refer to the pairs of types at equal percentiles $\{(x, s_m(x))|x \in [a_m, b_m]\}$ as the “efficient matching path.” We adopt the specific match value function $xy$ for analytical convenience. Since we allow the type distributions to be different for the two sides of the market, this specification is without loss of generality in so far as the match value function is multiplicatively separable and monotone in male and female types. To be precise, any match value function of the form $u(x)v(y)$, with $u$ and $v$ being positive-valued and monotone, can be transformed into the match value function $xy$ by redefining types and changing the distribution functions
appropriately.\footnote{When $u$ is monotonically increasing, the new type $\tilde{x}$ is $u(x)$; if it is monotonically decreasing, the new type $\tilde{x}$ is $1/u(x)$.} The separability assumption implies that each agent in a meeting place with pairwise random matching cares only about the average agent type on the other side, as opposed to the entire distribution. As a result, the monopolist problem of designing the sorting structure can be reduced to be a one-dimensional problem of match quality provision. The importance of this assumption will become clear in section 3. We will briefly discuss the case of non-separable match value functions in section 5.

A monopoly matchmaker, unable to observe types of men and women, can create a menu of meeting places with a pair of schedules of entrance fees $p_m$ and $p_w$. Each man or woman participates in only one meeting place. We will restrict each meeting place to have equal measure of men and women. We assume that men and women in each meeting place form pairwise matches randomly, with the probability of finding a match equal to 1 for all agents, and that the probability a type $x$ man meets a type $y$ woman is given by the density of type $y$ in that meeting place. In other words, the meeting technology in our model is random matching. For simplicity, we assume that meeting places cost nothing to organize. The objective function of the matchmaker is to maximize the sum of entrance fees collected from men and women.

The technology side of our framework is modeled on the motivating example of online dating. Imagine that each meeting place consists of two data bases, of men and women who have paid the corresponding subscription fees. Any man in the meeting place has access to the data base of women and can “search” it for a match. We have assumed that the probability of finding a match is 1 for all agents. This assumption rules out any size effect, which postulates a different probability of finding a match depending on the size of the market, and allows us to focus on the issue of price discrimination. The search technology in each meeting place, which is pairwise random matching, is admittedly primitive, compared to the actual matching technology used by online dating services where agents can search according to the information available on the data base and exchange further information through anonymous email correspondence. We have adopted the pairwise random matching technology in order to focus on the misrepresentation problem,
by implicitly assuming that any information volunteered by participants beyond what is signaled by their choices of meeting place is not credible and therefore cannot be used to improve matching efficiency. The importance and the implications of the assumption of random pairwise matching are discussed in section 5. Similarly, we have ignored the possibility of verifying certain information by providers of online dating services. For example, claims of college education in principle can be verified. Verifiable information can help the monopolist extract surplus. In the extreme case where the type information is public, the monopolist can achieve perfect discrimination through the perfect sorting. In general, the way availability of verifiable information changes the conditions for the optimality of the perfect sorting depends on how conditioning on public information affects the type distributions. We concentrate on unverifiable information and the consequent problem of misrepresentation.

We refer to a menu of meeting places as a sorting structure. Let \( \phi_m \) be a set-valued function that maps any male type \( x \) in \([a_m, b_m]\) to a subset \( \phi_m(x) \) of \([a_w, b_w]\). The set \( \phi_m(x) \) represents the set of female types that the male type \( x \) men can hope to meet. We sometimes refer to \( \phi_m(x) \) as type \( x \)'s "match set." We allow the possibility that male type \( x \) is excluded by the monopolist matchmaker, with \( \phi_m(\tilde{x}) = \emptyset \). Define \( \phi_w \) similarly, and denote \( \phi = \langle \phi_m, \phi_w \rangle \). For any \( X \subseteq [a_m, b_m] \), define

\[
\Phi_m(X) = \{ y | y \in \phi_m(x) \text{ for some } x \in X \}.
\]

That is, the female type set \( \Phi_m(X) \) represents the union of match sets of male types in \( X \). Define \( \Phi_w \) similarly.

**Definition 2.1.** A sorting structure \( \phi \) is feasible if for any \( x, \tilde{x} \in [a_m, b_m], y, \tilde{y} \in [a_w, b_w], X \subseteq [a_m, b_m] \) and \( Y \subseteq [a_w, b_w] \), i) \( y \in \phi_m(x) \) implies \( x \in \phi_w(y) \), and \( x \in \phi_w(y) \) implies \( y \in \phi_m(x) \); ii) \( \phi_m(x) \neq \phi_m(\tilde{x}) \) implies \( \phi_m(x) \cap \phi_m(\tilde{x}) = \emptyset \), and \( \phi_w(y) \neq \phi_w(\tilde{y}) \) implies \( \phi_w(y) \cap \phi_w(\tilde{y}) = \emptyset \); and iii) \( \Phi_m(X) \) has the same measure as \( \{ x | \phi_m(x) \subseteq \Phi_m(X) \} \), and \( \Phi_w(Y) \) has the same measure as \( \{ y | \phi_w(y) \subseteq \Phi_w(Y) \} \).

Condition i) is analogous to the standard symmetry condition for matching correspondences. It states that if type \( x \) men are participating in a meeting place where there
are type $y$ women, then type $y$ women are participating in a meeting place where there are type $x$ men, and vice versa. This condition is needed for a meeting place to have the interpretation of a matching market. Condition ii) requires that each type participates in at most one meeting place. This simplifies the analysis. Condition iii) requires that each meeting place consists of men and women of equal measures. This ensures that match probability is one for each agent in any meeting place. This condition helps us minimize the role of search technologies and focus on the impact of revenue-maximization on the sorting structure and matching efficiency.

3. Weak Sorting

The monopolist’s problem is to choose a sorting structure and the corresponding two fee schedules, one for males and one for females. A sorting structure assigns to each male type a set of potential female matches and to female types a set of potential male partners. The design problem appears multi-dimensional because what a type buys from the matchmaker is a type distribution on the other side of the meeting place. However, the assumption of a multiplicatively separable match value function allows us to reduce the problem to one dimension. Our first step of analysis is to substitute a pair of expected match types for each meeting place in the design problem, and transform the market design problem to a more familiar price discrimination problem.

A feasible sorting structure $\phi$ generates two schedules of expected match types, $q_m$ and $q_w$. The function $q_m : [a_m, b_m] \rightarrow [a_w, b_w] \cup \{0\}$ assigns to each male type the expected value of his match; the function $q_w : [a_w, b_w] \rightarrow [a_m, b_m] \cup \{0\}$ is the corresponding function for female types. We refer to $q = \langle q_m, q_w \rangle$ as a pair of “quality schedules.” Given $\phi$, we obtain $q_m$ and $q_w$ as

$$q_m(x) = \mathbb{E}[y | y \in \phi_m(x)];$$

$$q_w(y) = \mathbb{E}[x | x \in \phi_w(y)]$$

Without the restrictions of types participating in at most one meeting place, $\phi$ would not be sufficient to define $q_m$ and $q_w$ and we would need additional notation to specify the fraction of agents of a given type who participate in any given meeting place.
for all \( x \in [a_m, b_m] \) such that \( \phi_m(x) \neq \emptyset \) and \( y \in [a_w, b_w] \) such that \( \phi_w(y) \neq \emptyset \). We adopt the convention that if any type is excluded by the matchmaker, the match quality assignment is 0, which is the reservation utility. For notational simplicity all the lemmas in the remainder of this section refer to types that are served by the monopoly matchmaker. With the convention we have adopted, the lemmas can be easily restated to cover the excluded types.

As in a price discrimination problem, the monopolist does not observe agent types and must rely on self-selection of agents into their assigned expected match quality. Given equations (3.1), we can now formally state the optimal mechanism design problem of the matchmaker. Let \( p_m(x) \) be the participation fee for type \( x \) and define \( p_w(y) \) similarly; denote \( p = (p_m, p_w) \). The monopolist chooses a feasible sorting structure \( \phi \) and a pair of fee schedules \( p \) to maximize revenue

\[
\int_{a_m}^{b_m} p_m(x) \, dF(x) + \int_{a_w}^{b_w} p_w(y) \, dG(y),
\]

subject to incentive compatibility constraints

\[
xq_m(x) - p_m(x) \geq xq_m(\tilde{x}) - p_m(\tilde{x}),
\]
\[
yq_w(y) - p_w(y) \geq yq_w(\tilde{y}) - p_w(\tilde{y})
\]

for all \( x, \tilde{x} \in [a_m, b_m] \) and \( y, \tilde{y} \in [a_w, b_w] \) respectively, and participation constraints

\[
xq_m(x) - p_m(x) \geq 0,
\]
\[
yq_w(y) - p_w(y) \geq 0
\]

for all \( x \in [a_m, b_m] \) and \( y \in [a_w, b_w] \) respectively, where \( q \) is given in (3.1).

Under the complementarity assumed in the match value function, standard arguments imply that \( q_m \) being nondecreasing is necessary for the incentive compatibility constraints for men to be satisfied (see, e.g., Maskin and Riley, 1984). Further, the associated indirect utility \( U_m(x) \) of male type \( x \), defined as

\[
U_m(x) = xq_m(x) - p_m(x),
\]

Since the matching market is two-sided, self-selection involves a coordination problem that is absent in a standard price discrimination problem. We ignore such problem in this paper by assuming that the monopoly matchmaker can decide how agents self-select so long as the sorting structure is feasible and incentive compatible. See our companion paper (Damiano and Li, 2004) for a discussion of how to resolve the coordination problem.
satisfies the envelope condition
\[ U'_m(x) = q_m(x). \] (3.2)
at every \( x \) such that \( q_m(x) \) is continuous. Finally, condition (3.2) and the monotonicity of \( q_m \) together are sufficient for incentive compatibility. Similar observations hold for monotonicity of \( q_w \) and the indirect utility function \( U_w \) of women.

Unlike in a typical price discrimination problem, the monopolist can only choose schedules \( q_m \) and \( q_w \) consistent with some feasible sorting structure. Through a series of lemmas, we show how the feasibility constraints on the sorting structure translate into direct restrictions on quality schedules. Take a pair of nondecreasing quality schedules \( q \).

Monotonicity of the schedules leads to the following definition.\(^{11}\)

**Definition 3.1.** An interval \( T_m \subseteq [a_m, b_m] \) is a maximal pooling interval under \( q_m \) if \( q_m(\cdot) \) is constant on \( T_m \), and there is no interval \( T'_m \supset T_m \) such that \( q_m(\cdot) \) is constant on \( T'_m \).

Maximal pooling intervals \( T_w \) under \( q_w \) can be similarly defined. We say that \( q = \langle q_m, q_w \rangle \) is “feasible” if there is a feasible \( \phi = \langle \phi_m, \phi_w \rangle \) such that equations (3.1) are satisfied for almost all \( x \) and \( y \). We call \( \phi \) the “associated” sorting structure.

**Lemma 3.2.** If \( q \) is feasible, then for any maximal pooling interval \( T_m \) under \( q_m \) and any associated sorting structure \( \phi, \Phi_m(T_m) \) is a maximal pooling interval under \( q_w \).

**Proof.** Suppose \( \Phi_m(T_m) \) is not a maximal pooling interval under \( q_w \). There are two cases.

Case 1. Suppose that \( q_w \) is not constant on \( \Phi_m(T_m) \). Then, we can find \( y, \tilde{y} \in \Phi_m(T_m) \) such that \( q_w(y) < q_w(\tilde{y}) \). It follows from condition ii) in Definition 2.1 that \( \phi_w(y) \cap \phi_w(\tilde{y}) = \emptyset \) and \( \Phi_m(\phi_w(y)) \cap \Phi_m(\phi_w(\tilde{y})) = \emptyset \). Since \( T_m \) is a maximal pooling interval and \( y, \tilde{y} \in \Phi_m(T_m) \), we have
\[ E[t|t \in \Phi_m(\phi_w(y))] = E[t|t \in \Phi_m(\phi_w(\tilde{y}))], \]

\(^{11}\) There is no need to specify whether a maximal pooling interval contains the two end points. The assignment of values of \( q_m \) and \( q_w \) to the end points does not affect the revenue function stated later in Proposition 3.6.
which is possible only if
\[ \inf \Phi_m(\phi_w(\tilde{y})) < \sup \Phi_m(\phi_w(y)). \]

Then, there exist \( y_1 \in \Phi_m(\phi_w(y)) \) and \( \tilde{y}_1 \in \Phi_m(\phi_w(\tilde{y})) \) such that \( y_1 > \tilde{y}_1 \). It follows that \( q_w(y_1) = q_w(y) < q_w(\tilde{y}) = q_w(\tilde{y}_1) \), which contradicts the assumption that \( q_w \) is nondecreasing.

Case 2. Suppose that there is a \( W \supset \Phi_m(T_m) \) such that \( q_w \) is constant on \( W \). By a symmetric argument as in Case 1, we can show that \( q_m \) is constant on \( \Phi_w(W) \). Since \( W \supset \Phi_m(T_m) \), we can write
\[ \Phi_w(W) = \Phi_w(\Phi_m(T_m)) \cup \Phi_w(W \setminus \Phi_m(T_m)). \]

We claim that \( \Phi_w(\Phi_m(T_m)) \supseteq T_m \): if \( x \in T_m \), then there exists \( y \in \Phi_m(T_m) \) such that \( y \in \phi_m(x) \), which by condition i) of Definition 2.1 implies that \( x \in \phi_w(y) \), and therefore \( x \in \Phi_w(\Phi_m(T_m)) \). Further, \( \Phi_w(W \setminus \Phi_m(T_m)) \neq \emptyset \) because \( W \supset \Phi_m(T_m) \), and \( q_w \) is constant and different from 0 on \( W \). Finally, \( \Phi_w(W \setminus \Phi_m(T_m)) \cap T_m = \emptyset \), because \( y \not\in \Phi_m(T_m) \) implies that \( \phi_w(y) \cap T_m \neq \emptyset \) by condition i) of Definition 2.1. It follows that \( \Phi_w(W) \supset T_m \). Since \( q_m \) is constant over \( \Phi_w(W) \), we have reached a contradiction to the assumption that \( T_m \) is a maximal pooling interval under \( q_m \).

Q.E.D.

By symmetry, if a pair of nondecreasing schedules \( q \) is feasible, then \( \Phi_w(T_w) \) is a maximal pooling interval under \( q_m \) for any maximal pooling interval \( T_w \) under \( q_w \) and any associated sorting structure \( \phi \). A corollary of Lemma 3.2 is thus \( \Phi_w(\Phi_m(T_m)) = T_m \), and symmetrically \( \Phi_m(\Phi_w(T_w)) = T_w \). Another implication is that for any associated sorting structure \( \phi \), and for any maximal pooling interval \( T_m \) under \( q_m \), we have \( q_m(x) = \mathbb{E}[y|y \in \Phi_m(x)] \) for all \( x \in T_m \).

Symmetrically, for any maximal pooling interval \( T_w \) under \( q_w \) and for any \( y \in T_w \), we have \( q_w(y) = \mathbb{E}[x|x \in \Phi_w(T_w)] \).

Lemma 3.2 is the first step in showing that a pair of nondecreasing, feasible schedules \( q \) defines two sequences \( \{T_m^l\}_{l=1} \) and \( \{T_w^l\}_{l=1} \) of maximal pooling intervals in \([a_m, b_m]\) and

\[ \text{[12] In general, it is not true that } \phi_m(x) = \Phi_m(T_m) \text{ for all } x \in T_m, \text{ as there can be more than one way of assigning match sets for } x \text{ in } T_m \text{ such that } q_m(x) \text{ is constant. However, by condition iii) of Definition 2.1, we have } \int_{T_m} \mathbb{E}[y|y \in \phi_m(x)] \, dF(x)(F(T_m) - F(\inf(T_m))) \mathbb{E}[y|y \in \Phi_m(T_m)]. \text{ Since } \mathbb{E}[y|y \in \phi_m(x)] \text{ equals } q_m(x) \text{ and is constant on } T_m, \text{ it follows that } q_m(x) = \mathbb{E}[y|y \in \Phi_m(T_m)] \text{ for all } x \in T_m. \]
respectively, with $T^l_w = \Phi_m(T^l_m)$ and $T^l_m = \Phi_w(T^l_w)$ for each $l$. The next step is to identify the end points of each maximal pooling interval.

**Lemma 3.3.** If $q$ is feasible, then for any maximal pooling interval $T_m$ under $q_m$ and any associated sorting structure $\phi$, $s_m(\inf T_m) = \inf \Phi_m(T_m)$ and $s_m(\sup T_m) = \sup \Phi_m(T_m)$.

**Proof.** We first establish that if $x < \inf T_m$ then $\sup \phi_m(x) \leq \inf \Phi_m(T_m)$. To prove this claim by contradiction, suppose that there exists $y > \inf \Phi_m(T_m)$ such that $y \in \phi_m(x)$. Since $T_m$ is a maximal pooling interval and $x \not\in T_m$, we have $\phi_m(x) \cap \Phi_m(T_m) = \emptyset$. By Lemma 3.2, $\Phi_m(T_m)$ is an interval and hence $y \geq \sup \Phi_m(T_m)$. If $\inf \phi_m(x) \geq \sup \Phi_m(T_m)$, then since $q_m(x) = E[y|y \in \phi_m(x)]$ and $x < \inf T_m$, we have a contradiction to the assumption that $q_m$ is nondecreasing. If instead $\inf \phi_m(x) \leq \inf \Phi_m(T_m)$, then there exists $\tilde{y} \in \phi_m(x)$ such that $\tilde{y} \leq \inf \Phi_m(T_m)$. By condition ii) of Definition 2.1 and the definition of $q_w$ we have $q_w(y) = q_w(\tilde{y})$. Monotonicity of $q_w$ implies that $q_w$ is constant on $[\tilde{y}, y] \supset \Phi_m(T_m)$ therefore $\Phi_m(T_m)$ is not a maximal pooling interval under $q_w$, contradicting Lemma 3.2.

It follows from the above claim that $\phi_m(x) \subseteq \Phi_m([a_m, \inf T_m])$ for any $x < \inf T_m$, and hence $[a_w, \inf \Phi_m(T_m)] \supseteq \Phi_m([a_m, \inf T_m])$. Thus,

$$G(\inf \Phi_m(T_m)) \geq \int_{\Phi_m([a_m, \inf T_m])} dG.$$  

By condition iii) of Definition 2.1,

$$\int_{\Phi_m([a_m, \inf T_m])} dG \int_{\{x|\phi_m(x) \subseteq \Phi_m([a_m, \inf T_m])}\}} dF,$$

which implies that

$$G(\inf \Phi_m(T_m)) \geq F(\inf T_m).$$

By a symmetric argument, we have $\sup \phi_w(y) \leq \inf T_m$ for any $y < \inf \Phi_m(T_m)$. Hence, $[a_m, \inf T_m] \supseteq \Phi_w([a_w, \inf \Phi_m(T_m)])$ and

$$F(\inf T_m) \geq G(\inf \Phi_m(T_m)).$$

It follows that $G(\inf \Phi_m(T_m)) = F(\inf T_m)$, which by the definition of $s_m$ implies that $s_m(\inf T_m) = \inf \Phi_m(T_m)$.
The argument for \( s_m(\sup T_m) = \sup \Phi_m(T_m) \) is symmetric. \( \text{Q.E.D.} \)

Lemma 3.2 and Lemma 3.3 completely characterize a nondecreasing, feasible \( q \) for \( x \) and \( y \) in maximal pooling intervals. It remains to characterize \( q_m(x) \) and \( q_w(y) \) for any \( x \) and \( y \) not in a maximal pooling interval respectively.

**Lemma 3.4.** If \( q \) is feasible, then \( q_m(x) = s_m(x) \) for any \( x \in [a_m, b_m] \) such that \( x \) does not belong to any maximal pooling interval under \( q_m \).

**Proof.** Fix any sorting structure \( \phi \) associated with \( q \). We first show that, if \( x \) does not belong to any maximal pooling interval, \( \phi_m(x) \) is a singleton. Suppose \( y, \tilde{y} \in \phi_m(x) \) for some \( y < \tilde{y} \). By condition ii) of Definition 2.1, \( \phi_w(y) = \phi_w(\tilde{y}) \), and \( q_w(y) = q_w(\tilde{y}) \). Since \( q_w \) is monotone, it must be constant on the interval \([y, \tilde{y}]\). Therefore, there exists a maximal pooling interval \( W \supseteq [y, \tilde{y}] \). By construction, \( x \) belongs to \( \Phi_w(W) \) which is a maximal pooling interval by Lemma 3.2; a contradiction.

Let \( \phi_m(x) = \{ y_x \} \). Since \( q_m(x) = \mathbb{E}[y | y \in \phi_m(x)] \), we can write \( q_m(x) = y_x \). By monotonicity of \( q_m \), if \( \tilde{x} < x \) and \( \tilde{x} \) does not belong to a maximal pooling interval then \( y_\tilde{x} \leq y_x \) where \( \phi_m(\tilde{x}) = \{ y_\tilde{x} \} \). Together with Lemma 3.3, this implies that \( \Phi_m[a_m, x] \subseteq [a_w, \phi_m(x)] \). Clearly, \( y_x \) does not belong to a maximal pooling interval under \( \phi_w \) and \( \phi_w(y_x) = \{ x \} \), so an identical argument yields \( \Phi_w[a_w, y_x] \subseteq [a_m, x] \). Then, by condition iii) in Definition 2.1, we have \( F(x) = G(y_x) \), or \( q_m(x) = y_x = s_m(x) \). \( \text{Q.E.D.} \)

The following proposition summarizes the feasibility restrictions on incentive compatible quality schedules that we have derived from the restrictions imposed on feasible sorting structure (Definition 2.1). For the ease of notation, for any \( a_m \leq x < \tilde{x} \leq b_m \), let

\[
\mu_m(x, \tilde{x}) = \mathbb{E}[t | x \leq t \leq \tilde{x}]
\]

be the mean male type on the interval \([x, \tilde{x}]\). Define \( \mu_w(y, \tilde{y}) \) similarly.

**Proposition 3.5.** A pair of nondecreasing quality schedules \( \langle q_m, q_w \rangle \) is feasible if and only if i) for any maximal pooling interval \( T_m \) under \( q_m \) and each \( x \in T_m \), \( q_m(x) = \mu_w(s_m(\inf T_m), s_m(\sup T_m)) \) and \( q_w(s_m(x)) = \mu_m(\inf T_m, \sup T_m) \); and ii) for any \( x \) not in any maximal pooling interval under \( q_m \), \( q_m(x) = s_m(x) \) and \( q_w(s_m(x)) = x \).
PROOF. Necessity of i) and ii) follow immediately from Lemmas 3.2–3.4. For sufficiency, fix any $q$ that is nondecreasing and feasible. For each maximal pooling interval $T_m$ under $q_m$, construct the set-valued function $\phi_m$ such that $\phi_m(x) = [s_m(\inf T_m), s_m(\sup T_m)]$ for any $x$ in the closure of $T_m$, and correspondingly $\phi_w$ such that $\phi_w(y) = [\inf T_m, \sup T_m]$ for any $y \in [s_m(\inf T_m), s_m(\sup T_m)]$. For all other $x$, let $\phi_m(x) = \{s_m(x)\}$ and $\phi_w(s_m(x)) = \{x\}$. By conditions i) and ii) stated in the proposition, $\phi_m(x)$ and $\phi_w(y)$ are well-defined for all $x \in [a_m, b_m]$ and $y \in [a_w, b_w]$ respectively, and further, $\phi_m$ and $\phi_w$ satisfy equations (3.1) for almost all $x$ and $y$. Thus, $\langle q_m, q_w \rangle$ is feasible. \(Q.E.D.\)

The above result can be viewed as a characterization of any feasible sorting structure associated with an incentive compatible, feasible pair of quality schedules. We refer to the characterization as “weak sorting.” Since meeting places are mutually exclusive in type, in weak sorting if two types on the same side the market participate in two different meeting places, then the higher type not only has a higher average match type, but never gets a lower match.

We have completed transforming the design problem from choosing a feasible and incentive compatible sorting structure $\phi_m$ and $\phi_w$ to a problem of choosing a pair of non-decreasing quality schedules that satisfy Proposition 3.5. The advantage of this transformation is that from the mechanism design literature we know how to manipulate the incentive compatibility and individual rationality constraints associated with one-dimensional schedules $q$ to rewrite the matchmaker’s revenue. Define

$$J_m(x) = x - \frac{1 - F(x)}{f(x)};$$

$$J_w(y) = y - \frac{1 - G(y)}{g(y)}$$

to be the “virtual type” for male type $x$ and female type $y$ respectively. Virtual type of $x$ takes into account the incentive cost of eliciting private type information from type $x$. These are familiar definitions from the standard price discrimination literature (e.g., Myerson, 1981). Next, we combine the virtual types and define

$$K(x, y) = xJ_w(y) + yJ_m(x) \quad (3.3)$$
as the “virtual match value” for male type $x$ and female type $y$. Virtual match value of types $x$ and $y$ is based on the match value between $x$ and $y$ with proper adjustment of the incentive costs of eliciting truthful information from the two types.

For the following proposition, note that for any $q$ that is nondecreasing and feasible, there are at most countable many maximal pooling intervals. This is because for any maximal pooling interval $T_m$, the quality schedule $q_m$ is discontinuous at $\inf T_m$ (unless $\inf T_m = a_m$) and $\sup T_m$ (unless $\sup T_m = b_m$). Since $q_m$ is monotone, it can only have a countable number of discontinuities. Let $L$ be the total number of maximal pooling intervals under $q_m$; note that we allow $L$ to be infinite.

**Proposition 3.6.** Fix a pair of nondecreasing and feasible quality schedules $(q_m, q_w)$. Define $c_m = \inf \{x : q_m(x) > 0\}$, and let $\{T^l_m\}_{l=1}^L$ be the collection of all maximal pooling interval under $q_m$ over $[c_m, b_m]$. The maximum revenue generated by $(q_m, q_w)$ is given by:

$$
\int_{[c_m, b_m] \setminus \bigcup_{l=1}^L T^l_m} K(x, s_m(x)) \, dF(x) + \sum_{l=1}^L \int_{\inf T^l_m}^{\sup T^l_m} \int_{s_m(\inf T^l_m)}^{s_m(\sup T^l_m)} K(x, y) \, dG(y) \, dF(x) \quad (3.4)
$$

Thus, incentive compatibility and feasibility of the quality schedules imply that the monopoly matchmaker’s exclusion policy takes the form of a cutoff male type $c_m \in [a_m, b_m]$ such that male types $x < c_m$ and female types $y < s_m(c_m)$ are excluded. Given our characterization of incentive compatible, feasible quality schedules in Proposition 3.5, the proof of the above proposition follows the standard steps of integration by parts and application of the envelope conditions (3.2). More precisely, using the definition of the indirect utility function $U_m$, we can write the total revenue from the male side as

$$
\int_{c_m}^{b_m} (xq_m(x) - U_m(x)) \, dF(x),
$$
or

$$
\int_{c_m}^{b_m} xq_m(x) \, dF(x) + \int_{c_m}^{b_m} U_m(x) \, d(1 - F(x)).
$$

After integration by parts for the second term above, we can use (3.2) and the definition of virtual type function $J_m$ to rewrite the revenue from the male side as

$$
-U_m(c_m)(1 - F(c_m)) + \int_{c_m}^{b_m} q_m(x)J_m(x) \, dF(x).
$$
The revenue from the female side can be similarly stated. The cutoff types $c_m$ and $s_m(c_m)$ receive their reservation utility of 0 in any optimal price discrimination mechanism for the monopolist. The revenue formula (3.4) in Proposition 3.6 then follows from equation (3.3) and the characterization result of Proposition 3.5.

Proposition 3.6 restates the original sorting structure design problem given at the beginning of this section as choosing quality schedules $q$. We note that there are two components in the restated maximization problem: one is the exclusion policy or choosing $c_m$, and the other is the optimal sorting problem for a given $c_m$, which is the focus of the present paper. The necessary and sufficient conditions for the perfect sorting to be revenue-maximizing, derived in the next section, are a characterization of the optimal sorting problem.

The objective function of the optimal sorting problem is given by the revenue formula (3.4). The formula contains two terms, corresponding respectively to the revenue from the types that are perfectly sorted and the revenue from a sequence of pooling regions. The revenue from perfectly sorting the types in a region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ is the integral of the virtual match value function $K$ along the efficient matching path $\{(x, s_m(x)) | x \in (x_1, x_2)\}$, while the revenue from pooling the types in a region $T_m^l \times \Phi_m(T_m^l)$ is the integral of the virtual match value function over the entire region. Note that the quality schedule $q$ does not appear explicitly in the revenue formula; by Proposition 3.5, the feasibility constraint on $q$, together with the incentive compatibility constraint, has already pinned down $q$ once the sequence of maximal pooling intervals $\{T_m^l\}_{l=1}^{L}$ is given. Thus, the monopolist’s optimal sorting problem is reduced to choosing the break points of the maximal pooling intervals.$^{13}$ We can think of the monopolist partitioning the set of serviced male types $[c_m, b_m]$ into a sequence of pooling intervals and sorting intervals, with the set of serviced female types correspondingly partitioned. Since the revenue is written as sum of revenues from these intervals in the formula of Proposition 3.6, whether it is optimal to pool or to sort the types in one particular interval can be determined in isolation. This feature will be repeatedly used in the analysis of the next section.

$^{13}$ By definition two sorting intervals cannot be adjacent to each other. However, it is possible, and may even be optimal, to have two pooling intervals adjacent to each other.
4. Perfect Sorting

Proposition 3.5 establishes weak sorting as the outcome of satisfying both the incentive compatibility constraint in self-selection and the feasibility constraint on the sorting structure. Weak sorting can take different forms, from pooling the entire population of agents into a single meeting place to segregating each type into separate meeting places. Due to the assumption of complementarity in the match value function $x\gamma$, the finer the agents are partitioned into separate meeting places, the higher the matching efficiency in terms of the total match value.\(^{14}\) The question is whether the monopoly’s revenue is also increased.

In this section, we use the revenue formula of Proposition 3.6 to study the implications of revenue maximization to the sorting structure and matching efficiency. We are particularly interested in the perfect sorting structure. If the perfect sorting maximizes the monopolist’s revenue, then the monopolist has the same incentive to create meeting places as a social planner who maximizes the total match value. In this case, the incentive cost of eliciting private type information generates no distortion in terms of match quality provision. This is in contrast with the standard price discrimination result that quality is under-provided for all types but the very highest (Mussa and Rosen, 1978; Maskin and Riley, 1984). The standard result is commonly explained in terms of the tradeoff between “efficiency” and rent extraction: efficiency for a given type means a quality level that maximizes the trade surplus, defined as the type’s utility of consuming the quality less the cost of producing it, but the price function that implements the efficient quality schedule leaves too much rent to types. This tradeoff is resolved by a downward distortion of the quality level for all types except the highest. In contrast, in our model the tradeoff between efficiency and rent extraction is not resolved by the monopolist’s quality decisions, as these decisions are determined by the feasibility constraints (equations 3.1) once the sorting structure is chosen.

Before proceeding with the main analysis, we note that there are two aspects of the efficient matching. In order to maximize the total match value, the monopolist not only

\(^{14}\) Although this statement is intuitively obvious, we are not aware of a direct proof in the existing literature. Proposition 4.4 below provides such a proof. McAfee (2002) shows that a relatively large efficiency gain can be made by optimally splitting one market into two.
needs to create a continuum of meeting places to implement the perfect sorting, but also needs to have the same exclusion policy as the planner. By assumption, the match value of any pair of types is positive and the reservation utility of each type is zero, implying that the planner will have full market coverage (i.e., every type is matched by the planner.) In contrast, the virtual match value of a pair of types need not be positive, and so the monopolist may find it optimal to exclude some types. The focus of the present paper is when the revenue-maximizing sorting structure is the perfect sorting. This can be studied independently of the exclusion policy. The analysis below may be viewed as characterizing necessary and sufficient conditions for the optimality of the perfect sorting for any given exclusion policy.

4.1. Necessary condition

In a simple price-quality discrimination problem, where the trade surplus equals the product of the quality and the type less the cost of producing the quality, one presumes full separation of types, drops the monotonicity constraint on the quality schedule, and chooses a quality level to maximize the “virtual surplus” for each type, which is the trade surplus with the virtual type in place of the type. This pointwise maximization method cannot be applied in our mechanism design problem, because the quality schedules here are constrained by the choice of the feasible sorting structure through Proposition 3.5. For example, assuming full separation of types in our problem would uniquely determine the quality schedules as $q_m(x) = s_m(x)$ and $q_w(y) = s_w(y)$. Instead of pointwise maximization, we use a local approach to identify a necessary condition for the perfect sorting to be optimal. The idea is simple.\footnote{Bergemann and Pesendorfer (2001) use the same techniques to answer the question of how much private information an auctioneer should allow the bidder to learn about his valuation. The analogy between our sorting structure design problem and theirs can be seen if one thinks of a partition element in an information structure in their paper as a pooling of types in our problem. There are at least two important technical differences. First, in Bergemann and Pesendorfer types do not know who they are in a partitional element of an information structure and form expectations about their types, with all types in the element sharing the same expectation. This implies that the important incentive compatibility constraint is the one for the conditional average type. In our problem agents know their types, so the relevant incentive compatibility constraint for all types in a pooling meeting place is that for the lowest type. Second, we have an additional constraint in the revenue maximization problem that the quality schedules must be generated from a feasible sorting structure.} We start with the revenue from the perfect sorting
and study how it changes when we pool types in a small neighborhood around a point along the efficient matching path while keeping all other types separated. We then let the neighborhood become arbitrarily small.

Start with the quality schedules under the perfect sorting $s = (s_m, s_w)$, where $s_w$ denotes the inverse of $s_m$. For some $t \in (a_m, b_m)$ and a small positive $\epsilon$, construct a new pair of quality schedules $q(\epsilon)$ by pooling the male types on the interval $[t, t + \epsilon]$ with the female types on the corresponding interval $[s_m(t), s_m(t + \epsilon)]$, while retaining the perfect sorting structure outside the region $[t, t + \epsilon] \times [s_m(t), s_m(t + \epsilon)]$. Let $\Delta_t(\epsilon)$ be the revenue difference between $s$ and $q(\epsilon)$. We note that $q(\epsilon)$ is nondecreasing by construction, and feasible because it satisfies Proposition 3.5. Thus, we can apply the revenue formula of Proposition 3.6 and write $\Delta_t(\epsilon)$ as:

$$\int_t^{t+\epsilon} K(x, s_m(x)) \, dF(x) - \int_t^{t+\epsilon} \int_{s_m(t)}^{s_m(t+\epsilon)} K(x, y) \, dG(y) \, dF(x) \frac{F(t+\epsilon) - F(t)}{F(t+\epsilon) - F(t)}.$$ 

We will study the behavior of $\Delta_t(\epsilon)$ for $\epsilon \to 0$. In the following lemma, $K_{12}$ is the cross partial derivative of $K$. The proof is in the appendix.

**Lemma 4.1.** $\Delta_t(0) = \Delta'_t(0) = \Delta''_t(0) = 0$, and $\Delta'''_t(0) = \frac{1}{2} K_{12}(t, s_m(t)) f'(t) s'_m(t)$.

The next proposition follows immediately from the above lemma and gives a necessary condition for the perfect sorting to be optimal. This condition requires that the virtual match value function has nonnegative cross partial derivative along the efficient matching path $\{(x, s_m(x)) | x \in [a_m, b_m]\}$. For any $x \in (a_m, b_m)$, we say that $K$ satisfies “local path supermodularity” (local PS) at $x$ if $K_{12}(x, s_m(x)) \geq 0$. The relation between local PS and supermodularity will be discussed later. If local PS is not satisfied at some point on the efficient matching path, then the continuity of $\Delta_t(\epsilon)$ implies that there exists an $\epsilon > 0$ such that the monopoly matchmaker can increase revenue by pooling male types in $[t, t+\epsilon]$ and corresponding female types in $[s_m(t), s_m(t+\epsilon)]$ into a single meeting place, instead of perfectly sorting these types.

**Proposition 4.2.** Suppose that $K_{12}(x, s_m(x)) < 0$ for some $x \in (a_m, b_m)$. Then, any nondecreasing, feasible pair of quality schedules such that $x$ does not belong to the closure of a maximal pooling interval is non-optimal.
From the definition of virtual match value function $K$ (equation 3.3), we have

$$K_{12}(x, s_m(x)) = J_m'(x) + J_w'(s_m(x)).$$

(4.1)

In the special case where the two sides of the market have the same type distributions, with $s_m(x) = x$, the condition in Proposition 4.2 boils down to the virtual type function being nondecreasing. With asymmetric distributions, monotonicity of the virtual types on both male and female sides is not required, though for the perfect sorting to be optimal, at any point along the efficient matching path at least one side must have nondecreasing virtual type. Further, the local PS condition is not the same as the monotonicity of the sum of the virtual types along the efficient matching path. The latter is equivalent to

$$J_m'(x) + J_w'(s_m(x))s_m'(x) \geq 0,$$

so the sum of the virtual type functions may be decreasing at some $x$ while the virtual match value function satisfies local PS, and conversely the sum of the virtual type functions may be increasing while the virtual match value function fails local PS.

In the simple price discrimination problem where the utility of a type equals the product of the type and the assigned quality, monotonicity of the virtual type is necessary for a strictly increasing quality schedule to be optimal. Although in the special case where the two sides have the same type distribution our conclusion coincides with the standard monotonicity condition, the logic is different between the two models. In the price discrimination problem, monotonicity of the virtual type is equivalent to monotonicity of the solution to the pointwise maximization problem. Since the monotonicity constraint on the quality schedule is dropped in the pointwise maximization problem, monotonicity of the virtual type is necessary for the solution to be incentive compatible. In contrast, the necessity of the local PS condition follows from a variational argument over the revenue formula (equation 3.4), which respects the feasibility constraint as well as the monotonicity constraint on the quality schedules. Further, in the price discrimination problem, monotonicity of the virtual type is also sufficient for the solution to the pointwise maximization problem to be optimal. In the next subsection we explain why the local PS condition is generally insufficient for the perfect sorting to be optimal.
4.2. Sufficient conditions

By the proof of Proposition 4.2, failure of the local PS condition at any point on the efficient matching path implies that the monopoly matchmaker can increase the revenue by pooling adjacent types into a single meeting place. However, the local PS does not impose any constraint on the behavior of the virtual match value function away from a small neighborhood of the efficient matching path. As a result, it fails to ensure that a greater revenue cannot be generated by pooling a large set of types. For the perfect sorting to be optimal, the virtual match value function needs to satisfy a global version of the necessary condition in Proposition 4.2. We say that the virtual match value function $K$ satisfies “global” PS in the region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$, if for any $x, \tilde{x} \in (x_1, x_2)$,

$$K(x, s_m(x)) + K(\tilde{x}, s_m(\tilde{x})) \geq K(x, s_m(\tilde{x})) + K(\tilde{x}, s_m(x)).$$  (4.2)

When the above inequality holds with the strict sign we say that $K$ satisfies strict global PS. Clearly, global PS implies local PS of Proposition 4.2: letting $\tilde{x}$ converge to $x$ and $s_m(\tilde{x})$ converge to $s_m(x)$ in (4.2) and using the definition of cross partial derivatives imply that $K_{12}(x, s_m(x)) \geq 0$. However, global PS is weaker than requiring that $K$ be supermodular in the region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$. Supermodularity requires that for any $x, \tilde{x} \in (x_1, x_2)$ and $y, \tilde{y} \in (s_m(x_1), s_m(x_2))$ such that $x \leq \tilde{x}$ and $y \leq \tilde{y}$,

$$K(x, y) + K(\tilde{x}, \tilde{y}) \geq K(x, \tilde{y}) + K(\tilde{x}, y).$$

Instead, global PS only requires that the above inequality hold when $(x, y)$ and $(\tilde{x}, \tilde{y})$ are on the efficient matching path.

**Proposition 4.3.** If $K$ satisfies strict global PS in $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$, then any nondecreasing, feasible pair of quality schedules with $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ as the interior of a maximal pooling region is non-optimal.

\footnote{For general expositions of the concept of supermodularity and its economic applications, see Milgrom and Roberts (1990), Topkis (1998), and Vives (1990).}
The revenue from perfectly sorting the types in the region \((x, K)\) integral of the virtual match value function which is strictly positive because

\[ K \]

The first term on the right-hand-side of the above can be also written as

\[ \Delta = \frac{\int_{x_1}^{x_2} K(x, s_m(x)) \, dF(x)}{\int_{x_1}^{x_2} K(x, y) \, \frac{dG(y)}{F'(x) - F(x_1)}}. \quad (4.3) \]

The first term on the right-hand-side of the above can be also written as

\[ \int_{x_1}^{x_2} K(x, s_m(x)) \, dF(x) = \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(x, s_m(x)) \, \frac{dG(y)}{F'(x) - F(x_1)} \, dF(x). \]

With a change of variable \(y = s(x)\), we can also write

\[ \int_{x_1}^{x_2} K(x, s_m(x)) \, dF(x) = \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(s_w(y), y) \, \frac{dG(y)}{F'(x) - F(x_1)} \, dF(x). \]

In a similar way, after two changes of variable \(x = s_w(\tilde{y})\) and \(y = s_m(\tilde{x})\), the second term on the right-hand-side of (4.3) can be written as

\[ \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(x, y) \, \frac{dG(y)}{F'(x) - F(x_1)} \, dF(x) = \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(s_w(y), s_m(x)) \, \frac{dG(y)}{F'(x) - F(x_1)} \, dF(x). \]

Hence, \(\Delta\) is equal to one-half of

\[ \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} (K(x, s_m(x)) + K(s_w(y), y) - K(x, y) - K(s_w(y), s_m(x)) \, dF(x) \, dG(y) \, \frac{dG(y)}{F'(x) - F(x_1)} \],

which is strictly positive because \(K\) satisfies strict global PS in the range of the integration.

Q.E.D.

The idea of the above proposition comes from the revenue formula in Proposition 3.6. The revenue from perfectly sorting the types in the region \((x_1, x_2) \times (s_m(x_1), s_m(x_2))\) is the integral of the virtual match value function \(K\) along the segment of the efficient matching.
path \( \{(x, s_m(x)) | x \in (x_1, x_2)\} \), while the revenue from pooling the types is the integral of \( K \) over the entire region. By changes of variables we can write the revenue difference \( \Delta \) as one-half of the integral of the inequality (4.2) over the region.\(^\text{17}\)

Proposition 4.3 establishes that if the virtual match value function \( K \) satisfies global PS in \((x_1, x_2) \times (s_m(x_1), s_m(x_2))\), perfectly sorting types in that region does better than pooling the same types all together. This does not imply that the optimal sorting structure calls for the perfect sorting in the region, because it may be optimal to have a larger pool than \((x_1, x_2) \times (s_m(x_1), s_m(x_2))\). On the other hand, since by definition global PS in a region implies global PS in any subset of the region, if \( K \) satisfies global PS everywhere, then the perfect sorting is optimal. Thus global PS over \((c_m, b_m) \times (c_w, b_w)\) is a sufficient condition for the optimality of the perfect sorting.

When global PS is satisfied, the fee schedules can be easily calculated. Since

\[
p_m(x) = xs_m(x) - U_m(x),
\]

using condition (3.2) and the perfect sorting condition \( q_m(x) = s_m(x) \), we have

\[
p_m(x) = xs_m(x) - \int_{c_m}^{x} s_m(t) \, dt,
\]

where \( c_m \) is the cutoff type for the male side. A similar expression holds for the female fee schedule \( p_w \):

\[
p_w(y) = s_w(y)y - \int_{c_w}^{y} s_w(t) \, dt,
\]

Note that \( p_m \) and \( p_w \) are continuous. This property holds only when the perfect sorting is optimal. In general any pooling will make the quality schedule discontinuous. Since the indirect utility functions are necessarily continuous, the fee schedules will be discontinuous at the boundaries of each maximal pooling region. Note also that the total revenue from the two sides along the efficient matching path \((x, s_m(x))\) is

\[
p_m(x) + p_w(s_m(x)) = 2xs_m(x) - U_m(x) - U_w(s_m(x)).
\]

\(^{17}\) The proof of Proposition 4.3 also provides a direct argument of Proposition 4.2. If \( K_{12}(x, s_m(x)) < 0 \) for all \( x \in (x_1, x_2) \), then any nondecreasing, feasible pair of quality schedules that perfectly sorts types in any subset of the region \((x_1, x_2) \times (s_m(x_1), s_m(x_2))\) is non-optimal. We choose to present the original proof of Proposition 4.2 because it is constructive and illustrates the local nature of the necessary condition for the perfect sorting to be optimal.
This implies that the rate of increase of the total revenue is
\[ p'_m(x) + p'_w(s_m(x))s'_m(x) = s_m(x) + xs'_m(x), \]
which is one-half of the rate of increase of the total match value \(2xs_m(x)\).

Proposition 4.3 suggests that when the virtual match value function \(K\) satisfies global PS in some region \((x_1, x_2) \times (s_m(x_1), s_m(x_2))\), breaking up the region into sufficiently many small pooling regions generates more revenue than pooling all types in \((x_1, x_2) \times (s_m(x_1), s_m(x_2))\) together.\(^{18}\) However, global PS does not guarantee that dividing the market into meeting places always increases the revenue. The next proposition establishes that supermodularity of \(K\) is sufficient for this stronger result.\(^{19}\) This monotone convergence result is useful in practice because it implies that setting up a new meeting place always strictly increases revenue. It also illustrates a difference between supermodularity and global PS.

**Proposition 4.4.** Let \(q^*\) be a pair of nondecreasing, feasible quality schedules with \((x_1, x_2) \times (s_m(x_1), s_m(x_2))\) as the interior of a maximal pooling region. If \(K\) is (strictly) supermodular on \((x_1, x_2) \times (s_m(x_1), s_m(x_2))\), then for any \(t \in (x_1, x_2)\), any pair of nondecreasing, feasible quality schedules \(\hat{q}\) such that \(\hat{q}\) is identical to \(q^*\) outside \([x_1, x_2] \times [s_m(x_1), s_m(x_2)]\) and \(\hat{q}\) has \((x_1, t) \times (s_m(x_1), s_m(t))\) and \((t, x_2) \times (s_m(t), s_m(x_2))\) as the interiors of two maximal pooling regions generates (strictly) more revenue than \(q^*\).

**Proof.** Let the revenue difference between \(\hat{q}\) and \(q^*\) be \(\Delta\). Using the revenue formula from Proposition 3.6 we can show that \(\Delta\) is proportional to
\[
\int_{x_1}^{t} \int_{s_m(x_1)}^{s_m(t)} K(x, y) \, dF_l(x) \, dG_l(y) + \int_{t}^{x_2} \int_{s_m(t)}^{s_m(x_2)} K(x, y) \, dF_h(x) \, dG_h(y)
- \int_{x_1}^{t} \int_{s_m(x_1)}^{s_m(t)} K(x, y) \, dF_l(x) \, dG_h(y) - \int_{t}^{x_2} \int_{s_m(x_1)}^{s_m(t)} K(x, y) \, dF_h(x) \, dG_l(y), \tag{4.4}
\]

\(^{18}\) In an earlier version of the paper, we show that the revenue from perfectly sorting types in any region \((x_1, x_2) \times (s_m(x_1), s_m(x_2))\), can be approximated by breaking up the intervals \((x_1, x_2)\) and \((s_m(x_1), s_m(x_2))\) in a finite number of sufficiently small pooling intervals.

\(^{19}\) An implication is that increasing the number of meeting places always strictly increases match efficiency in terms of total match value. This can be seen by replacing \(K\) with the match value function \(xy\) in the proof of Proposition 4.4 below and noting that by assumption the match value function satisfies supermodularity.
where for any \( x \in (x_1, t) \) and \( \tilde{x} \in (t, x_2) \)

\[
F_l(x) = \frac{F(x)}{F(t) - F(x_1)}; \quad F_h(\tilde{x}) = \frac{F(\tilde{x})}{F(x_2) - F(t)},
\]

and for \( y \in (s_m(x_1), s_m(t)) \) and any \( \tilde{y} \in (s_m(t), s_m(x_2)) \)

\[
G_l(y) = \frac{G(y)}{F(t) - F(x_1)}; \quad G_h(\tilde{y}) = \frac{G(\tilde{y})}{F(x_2) - F(t)}.
\]

Next, apply the change of variables \( F_h(x) = F_l(\tilde{x}) \) to \( x \) in the second integral and in the fourth integral, and \( G_h(y) = G_l(\tilde{y}) \) to \( y \) in the second integral and in the third integral. Then, \( \Delta \) is proportional to

\[
\int_t^x \int_{s_m(x_1)}^{s_m(t)} \left( K(x, y) + K(F_h^{-1}(F_l(x)), G_h^{-1}(G_l(y)))
- K(x, G_h^{-1}(G_l(y))) - K(F_h^{-1}(F_l(x)), y) \right) \, dF_l(x) \, dG_l(y).
\]

The above is (strictly) positive because \( F_h^{-1}(F_l(x)) > x, G_h^{-1}(G_l(y)) > y \) and \( K \) is (strictly) supermodular on \((x_1, x_2) \times (s_m(x_1), s_m(x_2))\).

The idea behind Proposition 4.4 is to write the revenue difference \( \Delta \) between sorting the types in \((x_1, x_2) \times (s_m(t_1), s_m(t_2))\) into two meeting places and pooling all types in the region, as the integral of the inequality (4.2) where \( x \) varies between \( x_1 \) and \( t \) and correspondingly \( y \) between \( s_m(x_1) \) and \( s_m(t) \). This is achieved by a change of variables. Note that other changes of variables would also work. For example, one can define new integration variables by setting \( F_h(\tilde{x}) = 1 - F_l(x) \) and \( G_h(\tilde{y}) = 1 - G_l(y) \). The proof of the proposition proceeds in a similar fashion; the only difference is that for each \( x \in [x_1, t] \), inequality (4.2) applies to a different set of four points.

Equation (4.4) implies that a necessary condition for \((x_1, x_2) \times (s_m(x_1), s_m(x_2))\) to be the interior of a maximal pooling region is that there does not exist a point \((t, s_m(t))\) on the efficient matching path contained in the region such that the virtual match value function is “on average” supermodular at the point. An implication of this result is that if the matchmaker can create at least two meeting places, it would never be optimal to pool all men and women into a single market. This follows because regardless of the type
distributions, the virtual type functions $J_m$ and $J_w$ eventually become increasing towards the end of the efficient matching path and reach their respective maximum at the end. This in turn implies that there is always a point $(t, s_m(t))$ such that the virtual type functions satisfy

$$\min_{x \geq t} J_m(x) \geq \max_{x \leq t} J_m(x) \quad \text{and} \quad \min_{y \geq s_m(t)} J_w(y) \geq \max_{y \leq s_m(t)} J_w(y),$$

and therefore the virtual match value function is supermodular on average at $(t, s_m(t))$. At this point it would increase the revenue to split the market into two pools.

### 4.3. Path supermodularity and supermodularity

Supermodularity can be substantially stronger than path supermodularity. When the virtual match value function $K$ is twice differentiable, strict supermodularity of $K$ requires

$$\min_x J'_m(x) + \min_y J'_w(y) > 0,$$

while strict local PS requires

$$\min_x J'_m(x) + J'_w(s_m(x)) > 0.$$

Thus, when the minimum of $J'_m(x)$ and the minimum of $J'_w(y)$ are achieved at a point $(x, y)$ on the efficient matching path, the above two conditions coincide. This is the case when the efficient matching path $s_m(x)$ is linear, including the special case where $F$ and $G$ are identical so that $s_m(x) = x$. To see this, note that since $F(x) = G(s_m(x))$, we have

$$J'_w(s_m(x)) = J'_m(x) - \frac{1 - F(x)}{f(x)} \frac{s''_m(x)}{s'_m(x)}.$$

When $s_m$ is linear, if the minimum of $J'_m$ is achieved at some $x$, then $s_m(x)$ minimizes $J'_w$. In this case, there is no difference between supermodularity and local or global PS.

A linear efficient matching path occurs only when the distribution of type on one side of the market is the same as the distribution of a linear transformation of type on the other side. In general, however, the virtual match value function $K$ can satisfy local PS but fail to be supermodular. The following example illustrates the difference between the two concepts. Let $a_m = a_w = 0$ and $b_m = b_w = 1$, and fix $0 < \delta < 1$ and $0 < z < \frac{1}{2}$. Type
density functions are piece-wise linear:

\[ f(x) = \begin{cases} 
  \delta + \frac{2(1-\delta)}{z}(z-x) & \text{if } x \leq z \\
  \delta + \frac{2(1-\delta)}{1-z}(x-z) & \text{if } x > z;
\end{cases} \]

\[ g(y) = \begin{cases} 
  \delta + \frac{2(1-\delta)}{1-z}y & \text{if } y \leq 1-z \\
  \delta + \frac{2(1-\delta)}{z}(1-y) & \text{if } y > 1-z.
\end{cases} \]

Note that the density function of male type is bimodal with a trough at \( z \), while the density function of female type is single-peaked at \( 1-z \). See panel (a) of Figure 1. When \( \delta \) approaches 1, the two type distributions become identical (and uniform between 0 and 1) and the efficient matching path is the main diagonal in the type space. As \( \delta \) decreases, the type distributions become more dissimilar and the efficient matching path moves further above the diagonal for \( x < \frac{1}{2} \) and further below the diagonal for \( x > \frac{1}{2} \). See panel (b) of Figure 1. Note that \( s_m(z) < 1-z \). It is straightforward to verify the following properties of the virtual type functions: \( J_m'(x) < 0 \) for \( x \in [0, z) \), \( J_m' \) is discontinuous at \( z \) with \( \lim_{x \uparrow z} J_m'(x) < \lim_{x \downarrow z} J_m'(x) \), and \( J_m'(x) > 0 \), \( J_m''(x) < 0 \) for \( x \in (z, 1] \); \( J_w'(y) > 0 \), \( J_w''(y) < 0 \) for \( y \in [0, 1-z) \), \( J_w' \) is discontinuous at \( 1-z \) with \( \lim_{y \uparrow 1-z} J_w'(y) > \lim_{y \downarrow 1-z} J_w'(y) \), and \( J_w'(y) > 0 \) and \( J_w''(y) > 0 \) for \( y \in (1-z, 1] \). Given these properties, strict supermodularity of \( K \) is equivalent to

\[
\lim_{x \uparrow z} J_m'(x) + \lim_{y \uparrow 1-z} J_w'(y) > 0, \tag{4.5}
\]

while for local PS of \( K \)

\[
\lim_{x \uparrow z} J_m'(x) + J_w'(s_m(z)) > 0 \tag{4.6}
\]

is both necessary and sufficient. The latter claim follows from the fact that \( J_m'(x) + J_w'(s_m(x)) \) is decreasing for all \( x \in [0, z) \), and that \( J_m'(x) + J_w'(s_m(x)) > 0 \) for \( x \in (z, 1] \).\(^{21}\)

Moreover, since \( s_m(z) < 1-z \), \( J_w''(y) < 0 \) for \( y \in [0, 1-z) \), and \( \lim_{y \uparrow 1-z} J_w'(y) > \lim_{y \downarrow 1-z} J_w'(y) \), we have \( J_w'(s_m(z)) > \lim_{y \uparrow 1-z} J_w'(y) \) for any \( \delta \). When \( \delta \) is close to 1,

\(^{20}\) We have required the type density function \( f \) and \( g \) to be differentiable, but differentiability almost everywhere, which holds in the example, is sufficient for our analysis.

\(^{21}\) At \( x = s_w(1-z) \), \( J_w'(s_m(x)) \) is not defined but both the left and the right derivatives are positive.
both inequality (4.5) and inequality (4.6) are satisfied, because the two type distributions become uniform with linear increasing virtual types. When $\delta$ is close to 0, both inequality (4.5) and inequality (4.6) are violated, because $\lim_{x \to z} J'_m(x)$ decreases without bound while $J'_w$ is bounded. Since the left-hand-side of each inequality is continuous in $\delta$, there exists an intermediate values of $\delta$ such that $K$ is not supermodular but satisfies local PS.

An interesting issue is how changes in parameters of the model such as characteristics of the match value function and the type distributions affect the supermodular or PS properties of $K$. To explore this issue, consider monotone transformations of the type distribution on one side of the market, say, the male side. Let $\tau$ be an increasing, twice-differentiable function and denote $\tilde{x} = \tau(x)$. This transformation can be interpreted as a comparative statics exercise where the male type distribution changes from $F(x)$ to $\tilde{F}(\tilde{x})$ with $\tilde{F}(\cdot) = F(\tau^{-1}(\cdot))$ (and correspondingly the density function becomes $\tilde{f}(\cdot) = f(\tau^{-1}(\cdot))/\tau'(\tau^{-1}(\cdot))$) while the match value function remains $xy$. An equivalent interpretation is a comparative static exercise where the match value function changes from $xy$ to $\tau(x)y$ while the type distributions remain to be $F$ and $G$. If we adopt the first interpretation, then the derivative of the virtual male type function becomes

$$
\tilde{J}'_m(\tilde{x}) = J'_m(x) - \frac{1 - F(x)}{f(x)} \frac{\tau''(x)}{\tau'(x)}.
$$
where $x = \tau^{-1}(\tilde{x})$. Thus, when $\tau$ is linear with $\tau'' = 0$, the new virtual match value function has the same supermodular or local PS properties of the original $K$. These linear transformations generally change the variance of the type distribution on the male side; for example, the variance increases if the slope of $\tau$ is greater than 1. Thus, an implication is that changes in variances of type distributions do not necessarily make it more or less likely for the perfect sorting to be optimal. In contrast, when $\tau$ is convex (concave), the perfect sorting is less (more) likely to be optimal. Intuitively, a convex $\tau$ means that the new density function $\tilde{f}$, and hence the new virtual male type function, is less likely to be increasing (recall that $\tilde{f}(\cdot) = f(\tau^{-1}(\cdot))/\tau'(\tau^{-1}(\cdot))$). In the alternative interpretation of monotone transformations as changes in the match value function, we have that for fixed type distributions, all match value functions of the form $\tau_m(x)\tau_w(y)$ have the same supermodular and path supermodular properties for the virtual match value functions if $\tau_m$ and $\tau_w$ are linear. On the other hand, a change of the match value function from $xy$ to $x^2y$ would make it less likely for the perfect sorting to be optimal. Thus, the monopolist has less incentive to perfectly sort types when returns to scale in the match value function are higher.

5. Discussions

So far we have considered conditions for the perfect sorting to be optimal under two substantive assumptions about the reservation utility. First, we have assumed that the reservation utility is the same for the two sides of the market. This assumption can be easily dispensed without affecting the supermodularity and the path supermodularity properties of the virtual match value function. Given any exclusion policy (i.e. cutoff types $c_m$ and $c_w$ such that $c_w = s_m(c_m)$), the solution to the optimal sorting problem is independent of the reservation utilities, because the only change to the objective function (the revenue formula 3.4) is the addition of two constant terms $-U_m(1 - F(c_m))$ and $-U_w(1 - G(c_w))$, where $U_m$ and $U_w$ are the reservation utility for men and for women respectively. Thus, the conditions for the perfect sorting to be optimal will not change. Asymmetric reservation utilities will in general change the optimal exclusion policy. For example it might be optimal to charge a negative price to the cutoff type on the side with a higher reservation utility in order to induce greater participation and extract more rent from the other side.
The second assumption is that the reservation utility is type independent. However, higher types may have better outside options. This can be captured by assuming that men and women excluded from the monopolist’s mechanism can randomly match among each other for free. In this case the reservation utility of a type is the type’s expected payoff from joining the free pool, and is endogenously determined by the exclusion policy of the monopolist. Under any feasible, incentive compatible market structure, the types that participate in the free pool are determined by a cutoff rule, with only men and women below the respective cutoff types participating in the free pool. This is because the free pool corresponds to a participation fee of zero, so it cannot be optimal for the monopolist to create a meeting place with a quality lower than the quality of the free pool. Further, as in the case of exogenous type-independent reservation utility, the fees for the types served by the matchmaker are determined by the usual incentive compatibility constraints, rather than by the participation constraint that these types have to get as much utility from the matchmaker as from the free pool, even though higher types receive more utility from the free pool. This latter claim follows from the fact the indirect utility of a type \( x \) above the cutoff increases at the rate of \( q_m(x) \) (equation 3.2), while the utility from the free pool increases at the rate of the conditional mean of female types below the cutoff, which is lower than \( q_m(x) \). Thus, for any exclusion policy or a pair of cutoff types \( c_m \) and \( c_w \), the introduction of the free pool (with the utility for unmatched agents remaining zero) changes the objective function (the revenue formula 3.4) by adding two constants \(-c_m \mu_w(a_w, c_w)(1 - F(c_m))\) and \(-c_w \mu_m(a_m, c_m)(1 - G(c_w))\). This means that the solution to the optimal sorting problem does not change as a result of endogenous reservation utility, and the conditions for the perfect sorting to be optimal remain unchanged.\(^{23}\)

An important assumption in our model is that the match value function is multiplicatively separable. Without this assumption, the payoff to an agent from a random pairwise matching in a meeting place generally depends on the entire type distribution of

\(^{23}\) Endogenous reservation utilities will affect the monopolist’s exclusion policy. For example, when the type distributions are symmetric and the common virtual type function \( J(t) \) crosses zero only once, one can show that a free pool forces the matchmaker to increase market coverage. This follows because to counter the competition by the free pool, the matchmaker needs to admit more types at the bottom of the distribution so as to reduce the outside option for the participating types.
participants from the other side. This means that the monopolist problem of designing the sorting structure \( \phi \) cannot be reduced to be a one-dimensional problem of choosing quality schedules \( q \). In place of equations (3.1), the monopolist has to choose a pair of “match schedules” \( \alpha_m \) and \( \alpha_w \), with \( \alpha_m(x) \) representing the distribution of female types on the match set \( \phi_m(x) \) for male type \( x \). The key to the weak sorting result of Proposition 3.5 is the monotonicity condition on the quality schedule, but there is no counterpart to this ordering with a non-separable match value function because \( \alpha_m(x) \) is a multi-dimensional object. Thus the weak sorting result does not obtain, and we cannot further reduce the monopolist problem of designing the sorting structure to choosing the break points of maximal pooling intervals. However, if one is willing to assume weak sorting, then we can derive an analogous expression for the revenue formula of Proposition 3.6, and identify necessary and sufficient conditions for the perfect sorting to be optimal in the same way as in section 4.

More precisely, suppose that the match value function is \( v(x, y) \) with positive cross partial derivatives. Following standard arguments, we can show that \( \alpha_m(x)v_1(x, y) \) being nondecreasing is necessary for the incentive compatibility constraints for men, where \( \alpha_m(x)v_1(x, y) \) denotes the expectation with respect to \( \alpha_m(x) \) over \( y \in \phi_m(x) \) of \( v_1(x, y) \), the partial derivative of \( v \) with respect to the first argument. Note that this necessary condition is satisfied under weak sorting, because the match schedule \( \alpha_m(x) \) is ordered by first order stochastic dominance and \( v_1(x, y) \) is increasing in \( y \) by assumption. Further, if we define the indirect utility function \( U_m \) as

\[
U_m(x) = \alpha_m(x)v(x, y) - p_m(x),
\]

where \( \alpha_m(x)v(x, y) \) represents the expectation of \( v(x, y) \) with respect to \( \alpha_m(x) \) over \( y \in \phi_m(x) \), then \( U_m(x) \) satisfies the envelope condition

\[
U'_m(x) = \alpha_m(x)v_1(x, y),
\]

at every \( x \) such that the right-hand-side of the condition is continuous, which is everywhere except for the break points of the maximal pooling intervals.\(^{24}\) The envelope condition

\(^{24}\) For any \( x \) in a sorting interval, \( \alpha_m(x)v_1(x, y) \) is equal to \( v_1(x, s_m(x)) \), which is continuous in \( x \), while for any \( x \) in the interior of a maximal pooling interval \( T_m \), \( \alpha_m(x)v_1(x, y) \) is equal to the expected value of \( v_1(x, y) \) over \( y \) on \( \Phi_m(T_m) \), which is continuous on the interval \( T_m \).
and the monotonicity condition can be shown to be sufficient for incentive compatibility. Then, as in section 3, we obtain the same revenue formula as (3.4), with the virtual match value function $K$ redefined as

$$K(x, y) = v(x, y) - v_1(x, y) \frac{1 - F(x)}{f(x)} + v(x, y) - v_2(x, y) \frac{1 - G(y)}{g(y)}.$$  

The results on necessary and sufficient conditions for the optimality of the perfect sorting apply with no change.

An assumption complementary to multiplicative separability of the match value function is that agents are randomly matched within each meeting place. Without the assumption of random matching, the expected quality of a match in a meeting place may be type dependent and determined by the entire distribution of types in the meeting place. In this case, the expected payoffs from joining a meeting place would not be multiplicatively separable even if the match value function is, and this would create the same kind of analytical difficulties as discussed earlier. For example, if instead of one round of random matching we have sequential search as in Burdett and Coles (1997) or in Damiano, Li and Suen (forthcoming), the expected match quality for any type in a meeting place depends on which “class” the type belongs to. Moreover, the class structure is endogenously determined by the type distributions in the meeting place. How to incorporate sequential search into the framework of price discrimination is an interesting and challenging topic that deserves further research.

Appendix

A.1. Proof of Lemma 4.1

Proof. For notational convenience, denote $t + \epsilon$ as $\tilde{t}$. Define

$$R_t(\epsilon) = \int_{t}^{\tilde{t}} s_m(x) J_m(x) \, dF(x) + \int_{s_m(t)}^{s_m(\tilde{t})} s_w(y) J_w(y) \, dG(y);$$

$$\hat{R}_t(\epsilon) = \int_{t}^{\tilde{t}} \mu_w(s_m(t), s_m(\tilde{t})) J_m(x) \, dF(x) + \int_{s_m(t)}^{s_m(\tilde{t})} \mu_m(t, \tilde{t}) J_w(y) \, dG(y).$$
Then, \( \Delta_t(\epsilon) = R_t(\epsilon) - \hat{R}_t(\epsilon) \). Clearly, we have \( \Delta_t(0) = 0 \). We are going to take derivative with respect to \( \epsilon \) of \( R_t(\epsilon) \) and \( \hat{R}_t(\epsilon) \), and study the difference as \( \epsilon \) goes to zero.

For the first derivatives, let

\[ A_t(\epsilon) = s_m(\tilde{t})J_m(\tilde{t}) + (\tilde{t})J_w(s_m(\tilde{t})). \]

We have

\[ R'_t(\epsilon) = f(\tilde{t})A_t(\epsilon). \]

Next, we have

\[
\begin{align*}
\tilde{R}'_t(\epsilon) & = \mu_w(s_m(t), s_m(\tilde{t}))J_m(\tilde{t})f(\tilde{t}) + \mu_m(t, \tilde{t})J_w(s_m(\tilde{t}))g(s_m(\tilde{t}))s'_m(\tilde{t}) \\
& \hspace{1cm} + \mu'_w(s_m(t), s_m(\tilde{t}))s'_m(\tilde{t}) \int_t^{\tilde{t}} J_m(x) \, dF(x) + \mu'_m(t, \tilde{t}) \int_{s_m(t)}^{s_m(\tilde{t})} J_w(y) \, dG(y),
\end{align*}
\]

where \( \mu'_w(s_m(t), s_m(\tilde{t})) \) denotes the derivative of \( \mu_w(s_m(t), s_m(\tilde{t})) \) with respect to the second argument, and similarly \( \mu'_m(t, \tilde{t}) \) denotes the derivative of \( \mu_m(t, \tilde{t}) \) with respect to the second argument. Let

\[
\begin{align*}
\tilde{A}_t(\epsilon) & = \mu_w(s_m(t), s_m(\tilde{t}))J_m(\tilde{t}) + \mu_m(t, \tilde{t})J_w(s_m(\tilde{t})) \\
& \hspace{1cm} + (s_m(\tilde{t}) - \mu_w(s_m(t), s_m(\tilde{t})))J_m(t, \tilde{t}) + (\tilde{t} - \mu_m(t, \tilde{t}))J_w(s_m(t), s_m(\tilde{t})),
\end{align*}
\]

where \( J_m(t, \tilde{t}) = \int_t^{\tilde{t}} J_m(x) \, dF(x) \) and \( J_w(s_m(t), s_m(\tilde{t})) = \int_{s_m(t)}^{s_m(\tilde{t})} J_w(y) \, dG(y) \). We can write

\[ \tilde{R}'_t(\epsilon) = f(\tilde{t})\tilde{A}_t(\epsilon). \]

It is easy to see that \( A_t(0) = \tilde{A}_t(0) \); therefore, \( \Delta'_t(0) = 0 \).

Now consider the second derivative:

\[
\begin{align*}
R''_t(\epsilon) & = f'(\tilde{t})A_t(\epsilon) + f(\tilde{t})A'_t(\epsilon); \\
\tilde{R}''_t(\epsilon) & = f'(\tilde{t})\tilde{A}_t(\epsilon) + f(\tilde{t})\tilde{A}'_t(\epsilon).
\end{align*}
\]

Taking derivatives, we have

\[
A'_t(\epsilon) = s'_m(\tilde{t})J_m(\tilde{t}) + s_m(\tilde{t})J'_m(\tilde{t}) + J_w(s_m(\tilde{t}))(\tilde{t})J'_w(s_m(\tilde{t}))s'_m(\tilde{t}),
\]

and therefore

\[
A'_t(0) = s'_m(t)J_m(t) + s_m(t)J'_m(t) + J_w(s_m(t))(t)J'_w(s_m(t))s'_m(t).
\]
On the other hand, we have
\[ \hat{A}_t' (\epsilon) \mu'_w (s_m (t), s_m (\tilde{t})) s'_m (\tilde{t}) J_m (\tilde{t}) + J'_m (\tilde{t}) \mu_w (s_m (t), s_m (\tilde{t})) + \mu'_m (t, \tilde{t}) J_w (s_m (\tilde{t})) \\
+ \mu_m (t, \tilde{t}) J'_w (s_m (\tilde{t})) s'_m (\tilde{t}) + (s_m (\tilde{t}) - \mu_w (s_m (t), s_m (\tilde{t}))) J'_m (t, \tilde{t}) \\
+ (1 - \mu'_m (t, \tilde{t})) J_w (s_m (t), s_m (\tilde{t})) + (s'_m (\tilde{t}) - \mu'_w (s_m (t), s_m (\tilde{t}))) s'_m (\tilde{t}) J_m (t, \tilde{t}) \\
+ (\tilde{t} - \mu_m (t, \tilde{t})) J'_w (s_m (t), s_m (\tilde{t})) s'_m (\tilde{t}), \]
where \( J'_m (t, \tilde{t}) \) denotes the derivative of \( J_m (t, \tilde{t}) \) with respect to the second argument, and \( J'_w (s_m (t), s_m (\tilde{t})) \) similarly denotes the derivative of \( J_w (s_m (t), s_m (\tilde{t})) \) with respect to the second argument. Noting that
\[ \lim_{\epsilon \to 0} \mu'_m (t, \tilde{t}) = \lim_{\epsilon \to 0} \mu'_w (s_m (t), s_m (\tilde{t})) = \frac{1}{2}, \]
we obtain:
\[ \hat{A}_t' (0) = s'_m (t) J_m (t) + s_m (t) J'_m (t) + J_w (s_m (t)) + t J'_w (s_m (t)) s'_m (t). \]

Therefore, \( \Delta''_t (0) = 0. \)

We are led to compute the third derivatives of \( \Delta_t (\epsilon) \):
\[ R''_t (\epsilon) = f'' (\tilde{t}) A_t (\epsilon) + 2 f' (t + \epsilon) A_t' (\epsilon) + f (\tilde{t}) A_t'' (\tilde{t}); \]
\[ \hat{R}''_t (\epsilon) = f'' (\tilde{t}) \hat{A}_t (\epsilon) + 2 f' (\tilde{t}) \hat{A}_t' (\epsilon) + f (\tilde{t}) \hat{A}_t'' (\tilde{t}). \]

Taking derivatives and evaluating at \( \epsilon = 0 \), we have
\[ A''_t (0) = s''_m (t) J_m (t) + 2 s'_m (t) J'_m (t) + s_m (t) J''_m (t) + 2 J'_w (s_m (t)) s'_m (t) \\
+ t (J''_w (s_m (t)) (s'_m (t))^2 + J'_w (s_m (t)) s''_m (t)), \]
and
\[ \hat{A}''_t (0) s''_m (t) J_m (t) + s'_m (t) J'_m (t) + s'_m (t) J'_w (s_m (t)) + s_m (t) J''_m (t) \\
t (J''_w (s_m (t)) (s'_m (t))^2 + J'_w (s_m (t)) s''_m (t)) + \frac{1}{2} s'_m (t) (J'_m (t) + J'_w (s_m (t))). \]

Therefore,
\[ \Delta''''_t (0) = \frac{1}{2} (J'_m (t) + J'_w (s_m (t))) f (t) s'_m (t). \]

This completes the proof of the lemma. \( \text{Q.E.D.} \)
References


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