THE PRE-MARITAL INVESTMENT GAME

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Abstract. Two sides of a finite marriage market engage in costly investment and are then matched assortatively. The purpose of the investment is solely to improve the quality of the match that the trader can attain in the second stage. The paper studies the limits of equilibrium of these finite matching games as the number of traders gets large. It is shown that mixed Nash equilibria in the finite games converge to degenerate pure strategy equilibria in the limit in which both sides of the market invest too much.

1. Introduction

This paper studies the *pre-marital investment game.* A simple bilateral matching problem is augmented by a preliminary phase in which traders try to influence the outcome of the matching process by making costly 'investments' that make them more attractive as matching partners. This game provides a useful way to think about many applied matching problems. The one that motivated this research was the family matching problem studied in Peters and Siow (2002), and the closely related marriage problem Cole, Mailath, and Postlewaite (2001b) or Nosaka (2002) in which potential marriage partners or their families try to make themselves more attractive as partners by providing higher dowries, or more human capital. The marriage market operates in much the same way as a labor market where workers acquire costly human capital to distinguish themselves to firms who offer different wages and working conditions (for example Felli and Roberts (2000), Han (2002), Hopkins (2005), or Shi (1999)).[1]

The pre-marital investment game is a prototype for a broad variety of other bilateral matching problems. The early hedonic pricing literature, for example, studies a bilateral matching problem where different

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[1]The two sided nature of the investment distinguishes this paper from the literature concerned where investment occurs only on one side of the market - for example Bulow and Levin (2003).

firms produce output of different qualities and consumers have different tastes and amounts of money to spend (Rosen (1974) or Ekeland (2003)). Assortative matching pairs the consumer with the highest willingness to pay with the firm who produces the highest quality, and so on. Housing markets involve pre-match investments. Again, the seller of the house will make various improvements to try to match with a buyer who has a high willingness to pay. Discriminatory (double) auctions are also pre-marital investment games if the seller who makes the lowest offer is matched with the buyer who makes the highest bid.

These games are of interest because an investment that increases the quality of one's match partner partly internalizes the external benefits of investment. So, for example, it is typically argued that workers under-invest in human capital because part of the surplus associated with this investment is captured by the firm that hires them (the 'holdup problem'). Competition for partners will at least partly mitigate this without need for any special contractual arrangements or extensive form bargaining processes. Whether it does so fully is critical in regard to questions such as whether general education should be subsidized. Secondly, observed returns to investment are widely studied (for example the monetary returns to education are often estimated). These observed returns may or may not coincide with the return that a deviating trader might expect to receive. Understanding the difference between the two is important when making private investment decisions.

The purpose of this paper is to study perhaps the most natural model of pre-marital investment in which there are a finite number of traders on each side of the market. The first phase of the game involves investments, while the second involves a simple assortative matching of traders based on these investments. Assortative matching in the second stage is taken to be given, so equilibrium is just a Nash equilibrium of the finite player game.

Equilibria generally involve mixed strategies, and so are quite complex. The paper focuses on a special case and provides enough characterization of this equilibrium to show that these mixed equilibria become degenerate (i.e. pure) in the limit as the number of traders gets large. In the limit, investment levels on both sides of the market are too large. The market failure works the following way - workers, who are on the long side of the market, lose their investments if they fail to match. As other workers are randomizing in equilibrium, workers face some uncertainty about this, so they increase their investments to avoid unemployment. This, along with the randomness in workers investments creates and incentive for firms to raise their investments to compete for the best workers. The increase in firms investment makes employment even more attractive to workers who invest still more to ensure they are employed, and so on.

One consequence of this is that simple competition for partners does not 'resolve' the holdup problem as it does in Peters and Siow (2002) or Cole, Mailath, and Postlewaite (2001b) which are both essentially competitive models of the matching market. The non-cooperative analysis differs from the competitive analysis in two ways. The first, of course, is that competitive equilibria are efficient. Perhaps more fundamentally, the payoffs that traders expect to get when they deviate in the large non-cooperative game do not coincide with the payoffs that they expect when the deviate in the competitive equilibrium. Not only will these beliefs differ, but the expectations that workers and firms have about the implicit or hedonic 'value' of human capital will be different, and both will be different from the observed return to human capital.

The (weak) limit of equilibrium mixed strategies are not equilibrium strategies for the most natural definition of the limit game (though there are potentially many ways to define the limit game). These results together illustrate why analysis of a pre-marital investment game with a continuum of players can be misleading.

Equilibrium in pre-marital investment games are complex. This may explain why the literature on these games is small despite the enormous literature on matching without a prior investment phase. Payoff functions are discontinuous since only a tiny advantage over someone on the same side of the market will often ensure a match with a partner who has a considerably better quality. So Nash equilibria involve mixed strategies. There are a number of papers dealing with models in which investment occurs on only one side of the market. For example, Bulow and Levin (2003) study a finite 'market' in which firms compete in wages for workers with different, but exogenously fixed, productivities. They show that equilibrium mixed strategies lower wages relative to what they would be in a competitive market. Mixing results in inefficient mis-matching of workers and firms (though they illustrate that the costs associated with this aren't large). It is difficult to make a direct comparison between that paper and the paper here. They deal with a fixed finite number of workers and firms. Limits of such equilibria are of interest here. Since investments are fixed exogenously on one side of the market, inefficiencies only result from mismatches that occur because traders use mixed strategies. In the model discussed here, both sides invest, so it is the efficiency of the investment levels that are at issue.

A more closely related finite model is the paper by Felli and Roberts (2000). They allow workers to invest, then provide an ingeniously structured bidding game in which firms compete for workers. Their bidding game supports equilibrium in pure strategies. In their model investments are inefficiently low. Roughly speaking, workers who increase their investment are rewarded by the firm who hires them as if they had been hired by a less efficient firm. The reason is that the firm who hires them pays them just what they could have earned from their additional investment at their next best alternative. So a classic holdup problem arises for workers. Surprisingly, this problem does not disappear as the number of workers and firms increases. The reason is that the worst worker who is actually employed has unemployment as his next best alternative. Since investment is not productive in unemployment, he is not rewarded at all for incremental investment despite the fact that investment is profitable for the firm. Efficiency unravels from the bottom, much as it does here.

There are a couple of important differences between the the Felli and Roberts (2000) model and the one presented here. Transferable utility in their model means that the only utility that a worker gets from matching with a higher firm comes from the higher wage the firm pays. So the firm is able to capture all the surplus associated with additional worker investment. The holdup problem exists in its most extreme form. In the model here, workers derive an inalienable non-monetary return from matching with a better firm, which the firm cannot claim for itself. This stimulates investment and leads to the externality. In a mixed model with both monetary and non-monetary returns to matching these contrary incentives for investment would work against one another, so it is not clear what would happen. The second important difference is that their ex post bidding supports equilibrium in which traders use pure investment strategies. This eliminates the threat of unemployment that drives investment at the lower end of the distribution in the model here.

Hopkins (2005) analyzes a one sided investment problem with a continuum of traders. The investments in his paper are sunk as they are here, but the measures of traders on each side of the market are the same, so there is no pressure on traders to over-invest to ensure that they match. In the one sided investment case, this leads to equilibrium with very low levels of investment. With fixed investments on one side, this isn't really an efficiency issue. We discuss this in more detail below.

There are two other papers that provide a partially non-cooperative analysis of the pre-marital investment game. Peters and Siow (2002) analyze a non-cooperative equilibrium in a very simple example where workers' investments are also endogenous. They do not require that traders on the same side of the market be identical, but they only consider the case where there are two different kinds of traders on each side of the market.[2] It is too much to expect efficient investments with small numbers of traders simply because of standard strategic externalities that prevail in small games. So this paper doesn't address the limit problem that is of concern here. Nosaka (2002) studies a game with a continuum of traders on each side of the market, making a binary investment decision. The binary investment decision and the fact that traders have private information about their own costs make it difficult to address the efficiency problem that is the important concern in this paper. Further, as mentioned above, analysis of the game with a continuum of players can be misleading.

The complexities of equilibrium in the pre-marital investment game have prompted analysis using alternative solution concepts. Cole, Mailath, and Postlewaite (2001a), for example, study a cooperative game in which investments are followed by a matching game with transferable utility. When the number of traders is finite, this process supports efficient investments when a certain 'double overlapping' property holds. Double overlapping and transferable utility ensure that when a trader on one side of the market increases his or her investment, his partner's payoff is not affected. Efficiency is also supported in a limit version of this game Cole, Mailath, and Postlewaite (2001b).

Peters and Siow (2002) and Han (2002) study a competitive version of the pre-marital investment game without transfers, in which traders' investments are implicitly priced in terms of their partners' investment level. Their equilibrium also supports efficient investments. Yet, the competitive equilibrium appears to require that traders hold false beliefs about what will prevail out of equilibrium. Indeed, one of the objectives of this paper is to ask whether these out of equilibrium beliefs can be motivated by considering limits of Nash equilibria. Though the arguments below do not support the competitive solution per se, they do illustrate how to use Nash equilibria to support out of equilibrium beliefs in large games.

[2]The inefficiency of investments in that model can be attributed directly to the fact that there are a small number of players.

2. Fundamentals

For ease of discussion this paper simply adopts the worker-firm version of the problem.[3] The market consists of m 'firms' and n 'workers' with $n = \tau m$ for some constant $\tau > 1$.

Workers choose a human capital investment from a closed bounded interval $H \subset \mathbb{R}$. Firms choose a physical capital investment from a closed bounded interval $K \subset \mathbb{R}$.

Workers can be one of two possible types, *regular* or *poor*. A regular worker whose investment is h_i and who matches with a firm whose investment is k_j receives payoff

$$U(h_i, k_j) = u(h_i) + k_j$$

where u is assumed to be strictly concave, and to satisfy $u'(h^*) = 0$ for some strictly positive level of investment h^* in the interior of H. A regular worker who doesn't match has payoff $u(h_i)$. A poor worker always has 0 human capital investment. Every worker is a poor worker with probability λ which should be thought of as a small number.[45]

All firms have the same utility function and type. A firm whose investment is k_j and who matches with a regular worker whose human capital investment is h_i receives payoff

$$V(k_j, h_i) = v(k_j) + h_i$$

where v is assumed again to be strictly concave with $v'(k^*) = 0$ for some strictly positive investment k^* in the interior of the set K.[6] Any firm who doesn't match or who matches with a poor worker has payoff $v(k_i)$.

The investment by firms is intended to be something that is costly to the firm, but attractive to workers. One possibility would simply be to interpret it as a wage that firms offer, with the proviso that the firm's payoff is a non-linear function of its wage, which is 'paid' even if

[3] A previous working paper version of this paper includes an existence theorem for a very general version of the pre-marital investment game. The theorem uses the method of Reny (1999) to establish existence. This general proof is available at `http://microeconomics.ca/michael_peters/matching_limit_sim_add.pdf`.

[4] The existence of the poor worker ensures that the firm's payoff function satisfies a regularity condition used in the proof of existence.

[5] It is worth noting that investments are sunk so that a worker who doesn't match still pays the cost. The investment game then resembles an all pay auction or a tournament Rubinch-Pessach and Parreiras (2005).

[6] The assumption that h^* and k^* are strictly positive is just for convenience. If they are both zero and the marginal gain to additional investment on its own is negative, then exactly the same argument would hold provided 0 is not the efficient level of investment.

the firm doesn't match. These properties are consistent with the wage being an opportunity cost - for example, a capital investment the firm had to give up in order to offer the wage that it does.

In what follows, the investment levels h^* and k^* that satisfy $u'(h^*) = 0$ and $v'(k^*) = 0$ are referred to as the *bilateral Nash investment levels*. These are the investment levels that the firm and worker would choose on their own without worrying about the quality of their partner. Notice that the separability assumption ensures that these levels are independent of the quality of the trader's partner, provided this is fixed and independent of this investment.

Define investments

$$(\overline{h}, \overline{k}) \equiv \{(h, k) > (h^*, k^*) : v(k_j) + h_i = v(k^*); u(h_i) + k_j = u(h^*)\}.$$

These are the highest investments at which both firms and workers do as well as they would by themselves. It is assumed that $\overline{h}$ and $\overline{k}$ lie in the interior of the sets H and K respectively.

The properties of these utility functions are summarized in Figure 1. Worker investments are measured along the horizontal axis, firm investment on the vertical axis. Investment is costly for a worker, but initially, the marginal gains to investment exceed the costs so that utility is increasing in investment provided that the quality of the worker's partner is fixed. At the bilateral Nash level, the marginal benefit and cost of investment are equal, so a worker will invest no more unless the investment results in a better match. This sort of preference ordering leads to a family of indifference curves all of which look like the dark curve tangent to the horizontal axis at the point h^* in Figure 1. Higher indifference curves correspond to higher payoffs. The curve drawn corresponds to the payoff of a worker who doesn't match. Similar reasoning gives a family of indifference curves for firms that look like the dashed lines that become vertical at the level of investment k^* in the Figure. The bilateral Nash equilibrium is given by the point (h^*, k^*). Since the worker's indifference curve has to be perfectly flat at this point, while the firms has to be vertical, it is easy to see the standard underinvestment result (sometimes referred to as the 'holdup' problem). One efficient outcome is given by the point (h^e, k^e) in the Figure. The indifference curve for a firm who doesn't match is given by the dashed line tangent to the vertical axis at k^*.

For any vector $k \in \mathbb{R}^n$ let $k_{n:t}$ denote the t^{th} lowest element of k. Observe that the transformation $\{k_1, \ldots, k_n\} \to \{k_{n:1}, \ldots, k_{n:n}\}$ is continuous. Let $h = (h_1, \ldots, h_m)$ and $k = (k_1, \ldots k_n)$ be vectors of investments by workers and firms. The vectors h and k can also be written as (k_j, k_{-j}) and (h_i, h_{-i}) using a conventional notation. Define $\pi_l(k_j, k_{-j})$

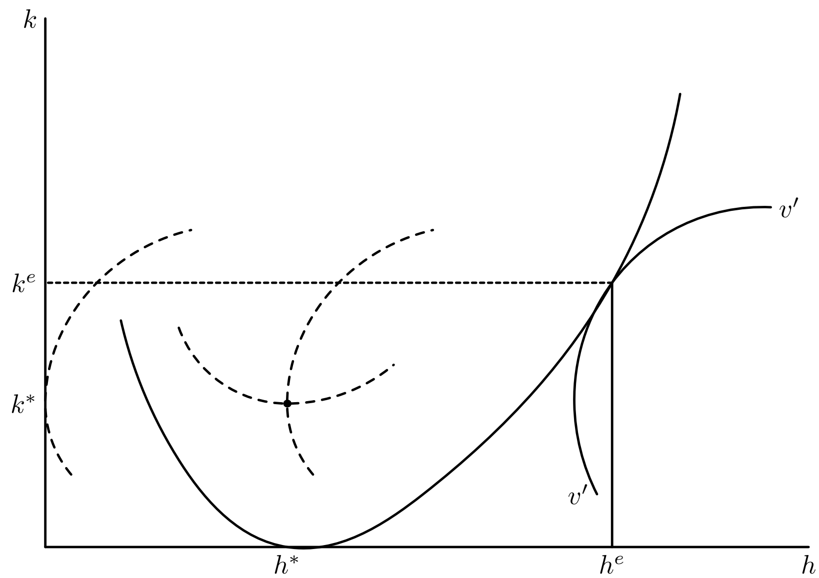

FIGURE 1. Properties of Preferences

to be the probability that firm j matches with the worker who has the l^{th} lowest investment among workers when workers and firms are matched assortatively and ties are resolved in each party's favor with equal probability. The expected payoff to the firm given investments (h,k) is given by

$$v(k_i) + \sum_{l=1}^{n} \pi_l(k_i, k_{-i}) h_{n:l}$$

(there are more workers than firms, so in this expression $\pi_l(k_i, k_{-i}) = 0$ for $l \leq n - m$). Observe that this payoff function is discontinuous because of the fact that whenever two firms have the same investment, one of them can discontinuously improve the quality of his partner by increasing investment slightly so that ties are resolved in his favour. Letting $\rho_l(h_i, h_{-i})$ be the corresponding probability for workers, the payoff to worker i is

$$u(h_i) + \sum_{l=1}^{m} \rho_l(h_i, h_{-i}) k_{m:l}$$

3. Equilibrium

Let F represent the common mixed investment strategy used by all firms (that is, $F(k_j)$ is the probability with which each firm j makes an investment less than or equal to k_j). Let $\tilde{G}$ describe the common mixed strategy for regular workers. With this formulation, the ex ante probability with which worker i invests h_i or less for $h_i \geq h^*$ is $G(h_i) = \lambda + (1-\lambda)\tilde{G}(h_i)$. The discussion that follows will be focused on G instead of $\tilde{G}$. Observe that with this interpretation, G always has an atom of size λ at the investment level 0.

The objective of this section is to characterize the *symmetric* Nash equilibrium of this simple pre-marital investment game as the solution to a pair of differential equations. The limit properties of the solutions to these equations are the objects of interest.

Lemma 3.1. *If symmetric equilibrium strategies G and F exist then they contain no atoms on the intervals $(h^*, \overline{h})$ or $[k^*, \overline{k}]$.*

Proof. In the symmetric case if all firms play an atom at some investment $k_j \geq k^*$ then there is a strictly positive probability that all firms will play k_j. Further, no matter what strategy workers are playing, there is a strictly positive probability (as long as $\lambda > 0$) that the number of poor workers will exceed $n-m$ so that some firm will be forced to match with a poor worker and receive zero benefit. On the other hand, every good worker will invest at least h^*. So by raising investment infinitesimally, any firm can discretely reduce the probability with which he is matched with a poor worker. This will discretely raise the firm's expected payoff in this event. Since matching is assortative, the quality of the firm's partner cannot fall in any event, so any loss associated with increasing investment can be made arbitrarily small. So this is a profitable deviation.

For workers, the argument is slightly simpler. If workers' strategy contains an atom at h_i, and all play h_i (which occurs with strictly positive probability) then $n-m$ of them will end up unmatched. By raising investment infinitesimally any worker can discretely increase the probability with which he matches in this event - a profitable deviation. □

The next Lemma is the analog of Lemma 2 in Hopkins (2005).

Lemma 3.2. *If symmetric equilibrium strategies G and F exist, then their supports are convex and contain h^* and k^*.*

Proof. To show that the supports of F and $\tilde{G}$ must contain the bilateral Nash investments, suppose the contrary. The argument is symmetric, so consider a firm who invests at the bottom of the support of F. There are no atoms in F by Lemma 3.1, so with probability 1 such an investment will result in a match with the worker whose investment is $n-m+1$ lowest since all other firms will have higher investments.

However, the same will be true if the firm cuts its investment to k^*. Since expected payoff for the firm is declining in k_i for $k_i > k^*$ for any given quality of the firm's partner, the firm can profitably deviate by cutting its investment.

The argument for convexity is similar. Suppose there is a gap $[k_0, k_1]$ in the support of firms' strategy. Let E_t be the event in which an investment k_1 is higher than exactly t investments by other firms (so the firm's partner will have the $n-m+1+t^{th}$ highest investment among workers). Since there are no atoms in F by Lemma 3.1, the probability of the event E_t will be exactly the same if the firm invests any $k' \in (k_0, k_1)$. Since $k_1 > k^*$ cutting investment will strictly increase the firm's expected profit in every event, without changing the probability that any event occurs. So cutting investment will constitute a profitable deviation. □

Lemma 3.3. *If symmetric equilibrium strategies G and F exist, then their supports are bounded above by $\overline{h}$ and $\overline{k}$.*

The proof is deferred to the appendix. Roughly, the proof follows from the fact that if $(h', k') \geq \left(\overline{h}, \overline{k}\right)$, then either $v(h') + k' < v(k^*)$ or $u(h') + k' < u(h^*)$.

The fact that there are no atoms in the supports of the firms' and regular workers' equilibrium strategies makes it possible to simplify the payoffs a little bit since we no longer need to worry about ties at any investment level in the support of their equilibrium strategies. Since workers and firms use mixed investment strategies, realized investments are random variables. Of course, these random variables depend on the distribution from which they are drawn. Since this distribution is always clear from the context, we write $\tilde{h}$ instead of $\tilde{h}^G$ to represent realized investments of workers when these investments are all drawn from the distribution G. With this notation, $\tilde{h}_{n:t}$ represents the realized value of the t^{th} lowest investment by workers.

Given F, let $\mathbb{E}_F \tilde{k}_{m:j}$ be the expected value of the j^{th} lowest order statistic when m draws are made from the distribution F. $\mathbb{E}_F \tilde{k}_{m:1}$ is then the expected value of the lowest order statistic, and so on. Reordering a vector is a continuous mapping. So the vector

$$\left\{\mathbb{E}_F \tilde{k}_{m:1}, \ldots, \mathbb{E}_F \tilde{k}_{m:m}\right\}$$

is continuous in F in the sense that if $F^\eta \to F$ weakly, then

$$\left\{\mathbb{E}_{F^\eta} \tilde{k}_{m:1}, \ldots, \mathbb{E}_{F^\eta} \tilde{k}_{m:m}\right\} \to \left\{\mathbb{E}_F \tilde{k}_{m:1}, \ldots, \mathbb{E}_F \tilde{k}_{m:m}\right\}$$

Similarly, $\left\{\mathbb{E}_G\tilde{h}_{n:1},\ldots,\mathbb{E}_G\tilde{h}_{n:n}\right\}$ represents the vector consisting of the expected values of the n order statistics generated when n draws are made from the distribution G.

Since ties occur with zero probability, a worker who invests h_i will match with a partner whose expected investment is

$$\sum_{t=n-m}^{n-1}\binom{n-1}{t}G^t(h_i)(1-G(h_i))^{n-1-t}\,\mathbb{E}_F\tilde{k}_{m:t-(n-m-1)}$$

(recall that $G(h)=\lambda+(1-\lambda)\tilde{G}(h)$). The similar expression for firms is

$$\sum_{t=0}^{m-1}\binom{m-1}{t}F^t(k_j)(1-F(k_j))^{m-1-t}\,\mathbb{E}_G\tilde{h}_{n:t+(n-m)+1}$$

The indices of the summation are different because firms always match (possibly with a poor worker), while workers may not.

By Lemma 3.2 it must be that in any symmetric equilibrium

$$(1)\quad v(k_j)+\sum_{t=0}^{m-1}\binom{m-1}{t}F^t(k_j)(1-F(k_j))^{n-1-t}\,\mathbb{E}_G\tilde{h}_{n:t+(n-m)+1}$$

is constant for every k in the support of F and

$$(2)\quad u(h_i)+\sum_{t=n-m}^{n-1}\binom{n-1}{t}G^t(h_i)(1-G(h_i))^{n-1-t}\,\mathbb{E}_F\tilde{k}_{m:t-(n-m-1)}$$

is constant for each h_i in the support of $\tilde{G}$. Furthermore, the supports of F and $\tilde{G}$ are both intervals whose lower bound is k^* and h^* respectively. The value of these payoffs can't be stated precisely except that firms' payoff must strictly exceed $v(k^*)$ because there are more workers than firms and all workers make strictly positive investment, and a regular workers' payoff must strictly exceed $u(h^*)$ because when the number of traders is finite, there is a strictly positive probability that all other workers will be poor workers.

If the payoffs are constant, the derivative of payoffs is uniformly zero. Focus first on workers. This means that

$$u'(h_i)+$$

$$\frac{dG(h_i)}{dh}\sum_{t=n-m}^{n-1}\binom{n-1}{t}G^{t-1}(h_i)(1-G(h_i))^{n-2-t}(t-G(h_i)(n-1))\,\mathbb{E}_F\tilde{k}_{m:t-(n-m-1)}$$

$$=0$$

and similarly for firms' payoffs.

This notation makes it possible to write the ordinary differential equation that characterizes the equilibrium strategy for workers as

$$\frac{dG(h_i)}{dh} =$$

(3)

$$\frac{-u'(h_i)}{\sum_{t=n-m}^{n-1} \binom{n-1}{t} G^{t-1}(h_i)(1-G(h_i))^{n-2-t}(t-G(h_i)(n-1))\,\mathbb{E}_F\tilde{k}_{m:t-(n-m-1)}}$$

with initial condition $G(h^*) = \lambda$.[7] A similar condition characterizes firms' equilibrium strategies, except that the initial condition is replaced with $F(k^*) = 0$.

A solution to (3) that has a strictly positive density on some subinterval $[h^*, h'] \subset H$ is called a *symmetric best reply* for workers to the distribution F.

Lemma 3.4. *Let F be a distribution function with support contained in the interval $[k^*, \overline{k}]$. Then there is a unique symmetric best reply G for workers to F. This best reply has convex support contained in $\overline{H} \equiv \{h_i \in H : \overline{h} \geq h_i \geq h^*\}$. This support contains the point h^*.*

The distribution F generates a vector of expected order statistics. Each element of this vector lies in the interval $[k^*, \overline{k}]$. If a solution exists, then it fixes workers' payoff at some level at least as high as $u(h^*)$, because a worker who invests h^* always matches with some probability. By the definition of $\overline{h}$, if $h_i > \overline{h}$, then the expression in (2) cannot provide this level of utility when $G(h_i) < 1$, because of the fact that the order statistics $\mathbb{E}_F\tilde{k}_{m:\cdot}$ are bounded above by $\overline{k}$. So if there is a solution, it must attain the value 1 somewhere on the interval $[h^*, \overline{h}]$. The rest of the tedious proof of this lemma simply involves checking that the appropriate Lipschitz condition holds. The details are included in the appendix. The property that is critical for the satisfaction of this Lipschitz condition is that a trader who is currently investing at the bottom of the support of any distribution function G can strictly increase the expected quality of her partner by increasing her 'rank' in the distribution of investments. For workers, this is ensured by the fact that unemployed workers have no partners. Firms at the bottom of the distribution will have partners anyway. This is where the assumption about poor workers is used. A firm at the bottom of the distribution of

[7] An equivalent characterization can be provided for the equilibrium strategy of the regular worker using the two definitions $G(h) = \lambda + (1-\lambda)\tilde{G}(h)$ and $G'(h) = (1-\lambda)\tilde{G}'(h)$. In that case the boundary condition is that $\tilde{G}(h^*) = 0$.

firms' investments will strictly lower the probability of being matched with a poor partner by increasing her rank.

So far, the pre-marital investment game is no more complicated than many other discontinuous games. If the vector of order statistics for investments for firms' investments were fixed exogenously (as in Bulow and Levin (2003) for example) then the analysis of equilibrium would be complete. The game is complicated by the fact that the vector of expected order statistics for firms' investments is endogenous. The point here is to show that symmetric equilibrium has undesirable efficiency properties. It isn't enough to assert that *if* such an equilibrium did exist, it wouldn't work well. So an existence theorem is needed. The existence theorem that follows provides one way to make this argument.[8]

Corollary 3.5. *The solution G to (3) varies continuously (in the sup norm) with the vector of order statistics* $\left\{\mathbb{E}_F\tilde{k}_{m:1},\ldots,\mathbb{E}_F\tilde{k}_{m:m}\right\}$.

Proof. The differential equation defined by (3) and the initial condition $G(h^*) = \lambda$ depends parametrically on the vector of order statistics $\left\{\mathbb{E}_F\tilde{k}_{m:1},\ldots,\mathbb{E}_F\tilde{k}_{m:m}\right\}$. The sup norm continuity of the solution in parameters is a standard consequence (for example Kreider, Kuller, and Ostberg (1968) Theorem 9-12 p 393.) of the Lipshitz condition established in the proof of Lemma 3.4. □

The existence theorem can now be proved using fixed point methods.

Theorem 3.6. *A symmetric Nash equilibrium for the premarital investment game exists.*

We give the proof explicitly since it is not hard. It illustrates the main technical complication in the pre-marital investment game. To find the equilibrium mixed strategies that workers use to make investments, it is necessary to know the expected returns to investment. These are endogenous and depend on the equilibrium mixed strategies used by firms, which in turn depend on workers' equilibrium strategies.

Proof. Let $\overline{K} = \{k_j \in K : \overline{k} \geq k_j \geq k^*\}$ and $\overline{H} = \{h_i \in H : \overline{h} \geq h_i \geq h^*\}$. Let F and G be two arbitrary distributions whose supports lie in $\overline{K}$ and $\overline{H}$ respectively. Use these to compute vectors of expected order statistics $\mathbb{E}\tilde{k}_{m:\cdot} \equiv \left\{\mathbb{E}_F\tilde{k}_{m:1},\ldots,\mathbb{E}_F\tilde{k}_{m:m}\right\}$ and

[8]A more detailed working paper version of this paper uses Reny's (Reny 1999) payoff security to establish existence of equilibrium in a more general version of the pre-marital investment game. The argument here makes heavy use of symmetry and the separability of payoffs.

$\mathbb{E}_G\tilde{h}_{n:\cdot} \equiv \left\{\mathbb{E}_G\tilde{h}_{n:1}, \ldots, \mathbb{E}_G\tilde{h}_{n:n}\right\}$ in $\overline{H}^n$ and $\overline{K}^m$. Let G' and F' be the unique symmetric best replies to G and F as given by Lemma 3.4. By that Lemma, the supports of G' and F'contain k^* and h^* and are contained in $\overline{H}$ and $\overline{K}$. So $\mathbb{E}_{F'}\tilde{k}_{m:\cdot}$ and $\mathbb{E}_{G'}\tilde{h}_{n:\cdot}$ are in $\overline{K}^m$ and $\overline{H}^n$respectively. Since $\overline{H}^n$ and $\overline{K}^m$ are both compact convex sets, the transformation $\left\{\mathbb{E}_F\tilde{k}_{m:\cdot}, \mathbb{E}_G\tilde{h}_{n:\cdot}\right\} \mapsto \left\{\mathbb{E}_{F'}\tilde{k}_{m:\cdot}, \mathbb{E}_{G'}\tilde{h}_{n:\cdot}\right\}$ maps a compact convex set into itself. So we establish that it has a fixed point. The solutions F' and G' are both sup norm continuous functions of the parameters $\left\{\mathbb{E}_F\tilde{k}_{m:1}, \ldots, \mathbb{E}_F\tilde{k}_{m:m}\right\}$ and $\left\{\mathbb{E}_G\tilde{h}_{n:1}, \ldots, \mathbb{E}_G\tilde{h}_{n:n}\right\}$ by Corollary 3.5. The operation of reordering a vector $x \in \mathbb{R}^n$ from lowest to highest is a continuous mapping, so its expectation is continuous in the sense that if $F^n \to F$ weakly, then $\left\{\mathbb{E}_{F^n}\tilde{k}_{m:1}, \ldots, \mathbb{E}_{F^n}\tilde{k}_{m:m}\right\} \to \left\{\mathbb{E}_F\tilde{k}_{m:1}, \ldots, \mathbb{E}_F\tilde{k}_{m:m}\right\}$. As sup norm convergence implies weak convergence the mapping $\left\{\mathbb{E}_F\tilde{k}_{m:\cdot}, \mathbb{E}_G\tilde{h}_{n:\cdot}\right\} \mapsto \left\{\mathbb{E}_{F'}\tilde{k}_{m:\cdot}, \mathbb{E}_{G'}\tilde{h}_{n:\cdot}\right\}$ is continuous in the usual sense. Hence by Kakutani's fixed point theorem, it has a fixed point. The strategies induced by this fixed point have convex support contained in $(\overline{H}, \overline{K})$ and contain the bilateral Nash points k^* and h^*since these properties are preserved by the transformation according to Lemma 3.4. If all others are using these strategies, then the payoff associated with using an investment in their support is constant. Since the strategies are atomless, deviating above their support yields the same quality partner as an investment at the top of the support, so deviations outside the supports are unprofitable. So the fixed point constitutes a Nash equilibrium. □

4. Inefficiency of Nash Equilibria in the limit

The objective in this section is to explore equilibrium when the number of traders is very large. The main results give conditions under which Nash equilibria 'converge' in an appropriate sense to pure strategy equilibria in which there is too much investment. Let (k^e, h^e) be the *efficient* investment levels[9] defined by the solution to the problem

$$\max v(k_j) + h_i$$

subject to

$$u(h_i) + k_j \geq u(h^*)$$

[9]There are many efficient investment levels. Attention is focussed here on the one where workers receive the payoff $u(h^*)$ because there are more workers than firms, and this is the only outcome that could be the limit of Nash equilibria.

In the solution to this problem, each trader must invest up to the point where the marginal gain that she conveys on her partner plus her own marginal gain is equal to her marginal cost of investment. So the efficient investment level will be strictly larger than the Nash level given by h^* and k^* respectively.

With a continuum of traders, an efficient outcome is achieved when the proportion $\lim_{m\to\infty} \frac{n-m}{n} = \frac{\tau-1}{\tau}$ of all workers invest h^* (or less if they are poor workers) and simply don't get partners. The rest all invest h^e. Firms should all invest k^e. Notice that this isn't a Nash equilibrium. The measure of workers who invest more than their bilateral Nash level must be the same as the measure of firms who invest in order that the outcome be efficient. Then any worker who cuts investment below h^e knows that he will continue to match with some seller as long as his investment remains above h^*. The issue here is whether this outcome can be sustained as the limit of Nash equilibria as the number of traders become large. The mixing that traders on both sides of the market do when there is a finite number of traders will deter deviations.

This paper shows that the equilibrium mixtures used by traders on both sides of the market converge weakly to step functions. For firms these strategies assign all probability in the limit to a single level of investment which we later refer to as k''. For workers, the strategies are somewhat more complicated. Mass λ is assigned to zero investment since any worker is a poor worker with probability λ. The distribution function associated with each worker's equilibrium strategy then jumps up to τ at h^*. So some regular workers invest their bilateral Nash levels. The probability is chosen so that the measure of workers who invest more than h^*in the limit is exactly equal to the measure of firms. Finally, each worker's equilibrium distribution jumps to 1 at a point h'' chosen such that $u(h'') + k'' = u(h^*)$.

The strong efficiency implications of this limit are derived from the fact that a firm who invests at the mass point h'' must achieve the same payoff as they would if they invested k^*. This implication of this can be understood with the help of Figure 2.

The efficient outcome involves investments h^e and $k^e = \kappa(h^e)$ where κ is the graph of workers' indifference curve through the efficient outcome. These objects are labeled in Figure 2. Efficiency requires that the indifference curves of workers and firms are tangent at $(h^e, \kappa(h^e))$. If this outcome happens to coincide with the limit of Nash equilibria, then when the number of traders is very large, firms should make investments close to $\kappa(h^e)$ most of the time. By Lemma 3.2, they should achieve

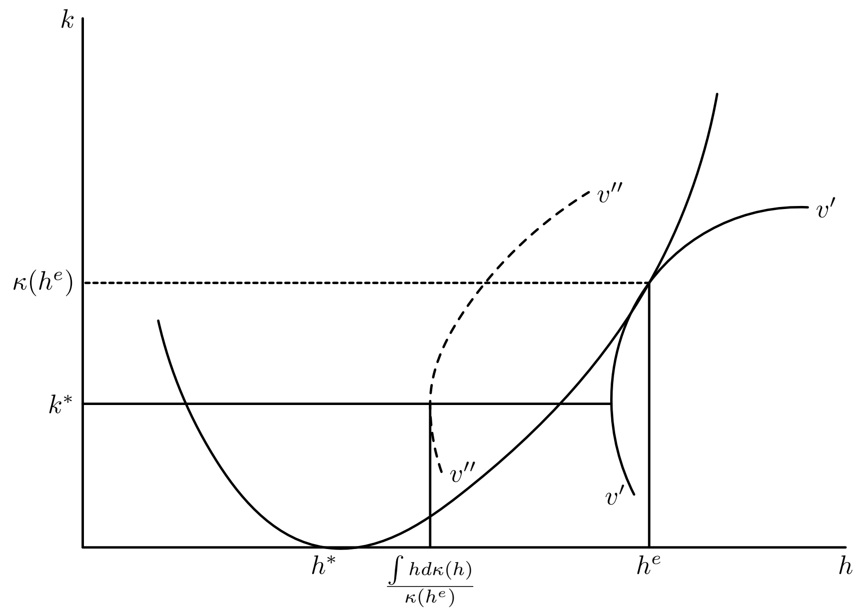

FIGURE 2. The Efficient Outcome Can't be Supported

exactly the same expected payoff by cutting their investments to the bilateral Nash level k^*. Since equality of payoffs in the support of the equilibrium strategy holds for every finite number of buyers and sellers, it also holds in the limit. So the expected quality of the firm's partner when it invests k^* must be equal in the limit to the length of the horizontal line from k^* out to the indifference curve for the firm through the efficient outcome $(h^e, \kappa(h^e))$. This line is drawn in Figure 2.

Surprisingly, it is possible to check whether this is true. Suppose for the moment that the equilibrium strategies converge as described above to step functions whose mass is concentrated at h'' and k''. A firm who makes the bilateral Nash investment k^* will have the lowest investment of all firms with probability 1 for every n by Lemma 3.1, and should therefore expect to match with the worker who has the lowest investment of all workers who actually find a match. This is the worker with the $n - m + 1^{st}$ lowest investment $\tilde{h}_{n:n-m+1}$ because there are $n - m$ more workers than firms.

It is possible to find an upper bound on the expectation of this random variable. Consider an arbitrary investment h' by a worker lying between h^* and the limiting mass point h'' of workers' investments. A worker who makes this investment will match with *some* firm if the $(n-m)^{th}$ lowest order statistic $\tilde{h}_{n-1:n-m}$ of the investments of the other workers is less than h'.[10] Let $\Pr\{\tilde{h}_{n-1:n-m} \leq h'\}$ be the probability of this event. This isn't the event the firm is interested in, but in the Nash equilibrium of the pre-marital investment game, the probability of this event is close to the probability that $\tilde{h}_{n:n-m+1}$ is less than or equal to h' when n and m are large, so we can use them interchangeably. If the event $\{\tilde{h}_{n:n-m+1} \leq h'\}$ occurs, then the worker will match with a firm whose expected investment is somewhere between k^* and k'' with very high probability. So the upper bound to the worker's payoff when he invests h' is $\Pr\{\tilde{h}_{n:n-m+1} \leq h'\} \cdot k''$ when n and m are large. The equilibrium payoff associated with the investment h', on the other hand, should be close to $\kappa(h')$, which is the expected quality he needs to attain his equilibrium payoff. This means that the distribution of the $n-m+1^{st}$ order statistic of workers' investments $\Pr\{\tilde{h}_{n:n-m+1} \leq h'\}$ is stochastically dominated by the distribution function $\frac{\kappa(h')}{\kappa(h'')}$ when m and n are large.

This is just the information needed to compute the upper bound on the expected quality of the deviating firm's partner. Just integrate using this distribution function to get the upper bound $\int_{h^*}^{h''} \tilde{h} \frac{d\kappa(\tilde{h})}{\kappa(h'')}$. Since $\kappa(\cdot)$ is the graph of a worker's indifference curve through the point $(h^*, 0)$, it isn't too hard to see (by inspection of Figure 2 for example) that the point $(\int_{h^*}^{h^e} \tilde{h} \frac{d\kappa(\tilde{h})}{\kappa(h^e)}, k^*)$ will lie to the left of the firm's indifference curve through the efficient outcome (h^e, k^e) under reasonable conditions. To make the theorems that follow tighter, we impose the following (stronger) restriction on u and v.

Condition 4.1. Let $\kappa(h)$ be the solution to $u(h) + k = u(h^*)$. Then $u\left(\int_{h^*}^{h^e} \tilde{h} \frac{d\kappa(\tilde{h})}{\kappa(h^e)}\right) + k^* > u(h^*)$.

This argument can be formalized to give the following characterization result:

Proposition 4.2. *Suppose Condition 4.1 holds. Let G and F be weak limits of workers' and firms' equilibrium strategies along some subsequence of equilibria in which $m \to \infty$. Then there is a pair (h'', k'') such that $h'' > h^e$, $k'' > k^e$;*

$$G(h_i) = \begin{cases} \lambda & \text{if } h_i < h^* \\ \frac{\tau-1}{\tau} & \text{if } h^* \leq h_i < h'' \\ 1 & \text{otherwise} \end{cases} ;$$

[10] We don't have to worry about equality by Lemma 3.1.

$$F(k_j) = \begin{cases} 0 & if\ k_j < k'' \\ 1 & otherwise \end{cases};$$

and

$$v(k^*) + \frac{1}{\kappa(h'')} \int_{h^*}^{h''} \tilde{h} d\kappa\left(\tilde{h}\right) \geq v(k'') + h''.$$

This outcome is not obviously an equilibrium of the limit game in which there are a continuum of workers and firms. It would be natural for any worker investing at the mass point h'' to assume that he could cut his investment without losing his existing partner. That is because the measure of workers who invest h'' is the same as the measure of firms who invest k'' in the limit. If he cuts his investment, the firm he will match with will be mad, but there will be no better worker for him to match with since they are already paired up with other firms.[11]

The large finite games show why this reasoning is inappropriate. When the number of traders is finite every worker invests close to this mass point with very high probability - *but this probability is not equal to* 1 by Lemma 3.1. The same is true for firms. So a worker or firm who invests close to this mass point will strictly out invest a number of traders on his own side of the market with high probability (though the probability that he strictly out invests any *given* trader will still be infinitesimal). Furthermore, the worst traders on the other side of the market will actually have investments close to the bilateral Nash level (even though, again, the probability that any given trader has such an investment is infinitesimal). So when a worker cuts his investment below this mass point, he should find that the expected quality of his partner falls. This *must* be true to support the Nash equilibrium. This logic simply can't be seen by looking at the limit game.

The Proposition also establishes that workers' and firms' investments will be too high. The over-investment result is the consequence of two interacting externalities. A worker, for example, who invests more creates a negative externality for other workers, since workers are on the long side of the market. The better one worker looks, the more likely it is that other workers will end up without partners. This spurs intense competition by workers in human capital aimed at trying to

[11]The limit game admits a number of alternative definitions. One possibility working in the continuum is simply to observe that two sets that have the same measure are not necessarily isomorphic. So when the worker cuts his investment, there will simply be some other worker there to replace him. The difficulty with this definition is that it supports every pair (h_i, k_j) that gives workers payoff $u(h_i) + k_j = u(h^*)$ as an equilibrium of the limit game. There may be more than one limit of Nash equilibria, but the limiting procedure is considerably more restrictive than this.

avoid remaining unmatched. This tends to benefit firms who end up with better partners. However, firms are faced with a similar problem, the better one firm looks, the more likely another firm will end up matched with a poor worker. So firms invest more. This makes the difference in payoff for a matched and unmatched worker even higher, leading to more investment in human capital, and so on. All this drives investment above the efficient level.

5. Random Outcomes

An alternative[12] way to illustrate the difficulties that arise in the pre-marital investment game is to assume that the outcome of traders' investments are random. A complete analysis of this case goes far beyond the scope of this paper, but it is possible to illustrate why the results here will be robust to this change in specification.

For illustration we proceed directly to the limit game and use a very simple formulation for uncertainty. Suppose that every investment is uncertain in the specific sense that investment h by a worker yields an investment outcome $h + \epsilon$ where ϵ is uniformly distributed on the interval $[-a, a]$. Make exactly the same assumption for firms.

First consider the possibility that firms and workers might invest efficiently. For this purpose, we can imagine that the measure of firms who invest is equal to the measure of workers who invest. We will force the payoff to investing workers to be equal to $u(h^*)$ to support equilibrium. Suppose that firms and workers who do invest make investments k^0 and h^0 respectively.

Arguing as if the law of large numbers holds in the continuum, realized investments will be uniformly distributed between $h^0 - a$ and $h^0 + a$ for workers, and $k^0 - a$ and $k^0 + a$ for firms. The ex ante expected payoff to a worker who invests h from assortative matching is then

$$u(h) + \int_{-a}^{-a+h^0-h} (k^0 - a) d\frac{\epsilon}{2a} + \int_{-a+h^0-h}^{a} (h + \epsilon + k^0 - h^0)) \frac{d\epsilon}{2a}$$

if $h \leq h^0$, and

$$u(h) + \int_{-a}^{a-(h-h^0)} (h + \epsilon + k^0 - h^0) \frac{d\epsilon}{2a} + \int_{a-(h-h^0)}^{a} (k^0 + a) \frac{d\epsilon}{2a}$$

[12] Suggested by Jeroen Swinkels and by a referee of a previous version of this paper.

otherwise. From the first order conditions for optimization of these functions, a necessary condition for the pair (h^e, k^e) to be an equilibrium is that $u'(h) = v'(k) = 1$, which is exactly the condition for efficient investment.

Since the workers who choose not to invest must not want to deviate, $u(h^e) + k^e = u(h^*)$. To see the problem with this, suppose that a is small, and that a worker deviates to an investment slightly larger than $h^* + a$. Then the measure of workers who invest more than $h^*$is still equal to the measure of firms who invest k^e, so the deviating worker will retain a partner whose quality is $k^e - a$ with probability 1. If a is small enough, this will be a profitable deviation.[13]

To support equilibrium, the common investments of firms and workers must then be close to their Nash levels to prevent this kind of deviation (for example, this kind of deviation won't work if $h^* > h^0 - a$). Such low levels of investment can't be equilibria because of the marginal conditions given above, the marginal gain to investment is close to one, while the marginal cost is close to zero when investment is near its Nash level. So equilibria will be in mixed strategies when a is small.

It would be of some interest to see whether equilibrium outcomes when a is small are close to those when a is zero. As the analysis in the main body of the paper shows, the limit game itself doesn't provide much insight into this problem, it is important to study the limits of equilibria from finite matching problems in order to understand the expected benefits and costs of deviations. This evidently goes well beyond the scope of this paper.

The over-investment result also extends to the random outcome. One way to eliminate the profitable deviation mentioned above is to focus on a completely symmetric equilibrium in which all workers invest even though many of them will never match. Then a worker who deviates by investing below the support of the equilibrium strategies as above, will be displaced and lose his or her partner.

This kind of equilibrium has to involve excessive investment since many of the workers who invest won't find partners, so their investment will be wasted. Furthermore, it isn't clear whether there aren't Pareto dominating equilibria where firms and workers randomize about whether or not to invest as they do in the model with certain investment outcomes. Nonetheless, this equilibrium illustrates the second source of inefficiency in the pre-marital investment game.

[13] This issue is not due to the fact that the support of the distribution of outcomes is shrinking. If the support stays large while the mass of the distribution becomes more concentrated, the same kind of deviation would be profitable.

To see it, suppose again that the measure of the set of firms is 1, while the measure of the set of workers is $\tau > 1$. Then in the symmetric equilibrium a set of workers of measure $\frac{\tau-1}{\tau}$ will end up investing, but not matching. Assuming that workers and firms commonly invest h^0 and k^0 respectively as above, realized investment outcomes will be uniformly distributed on $[h^0 - a, h^0 + a]$ for workers and $[k^0 - a, k^0 + a]$ for firms. Only those workers who realized investment outcomes are above h^+ satisfying $\frac{h^0+a-h^+}{2a}\tau = 1$ or $h^+ = h^0 + a\frac{\tau-2}{\tau}$ will actually succeed in finding a partner. Then using assortative matching against a uniform distribution of firms' investments on $[k^0 - a, k^0 + a]$ yields an expected payoff for workers who invest at least h^0 given by the following expression:

$$u(h) + \int_{h^e-h+\frac{a(\tau-2)}{\tau}}^{h^e+a-h} (\tau(h+\epsilon) + K)\frac{d\epsilon}{2a} + \int_{h^e+a-h}^{a} (k^e + a)\frac{d\epsilon}{2a}$$

where K is a constant. If $\epsilon$is too low, the worker won't match at all. The worker can reduce the probability that this event occurs by raising his investment. This is the reason the lower limit on the first integral depends negatively on the deviating worker's investment h. Then, for each additional dollar the worker invests, the quality of his partner rises by τ dollars. Hence the marginal benefit of investment has increased from 1 in the asymmetric case discussed above, to something considerably larger than 1. These are the two forces that drive workers' investments above the efficient level which occurs when $u'(h^e) = 1$.

The nature of equilibrium here results from two forces - the fact that there are more workers than firms, and the fact that investments are endogenous on both sides of the market. To take an extreme case, suppose that the measures of the set of workers and firms is the same and that firms' investments are exogenously set at their efficient level k^e. This is a very special case of the model consider by Hopkins (2005). There is only one equilibrium outcome in the continuous case where all workers make their Nash investments. This simply follows from the fact that all workers know they will match. If the firms' investments are truly exogenous, then this outcome is Pareto efficient. In the special case of identical traders and equal measures, it is possible to support an outcome where investments are too low by making firms' investments endogenous, then having them make their Nash investments as well. No one has an incentive to deviate because they can't improve the quality of their partner.

This outcome breaks down when there are more workers than firms, since workers will start to compete to avoid unemployment (assuming that workers prefer to match with a firm who has made its Nash investment than to remain unemployed). The dispersion of worker investments that this generates will cause firms to compete for the better workers, ultimately leading to the excessive investment as explained above.

6. Other Variants

One obvious shortcoming of the model presented above is that traders on the same side of the market have identical preferences. To extend the model to allow for different investment costs would involve non-trivial technical problems, mostly because the differential approach outlined above would fail. A very simple version of a model with different investment costs appears in Peters and Siow (2002) which analyzes the case in which there are two firms with different investment costs, and two workers with different investment costs. The equilibrium still involves mixed strategies, but the supports of these strategies differ across firms and workers, and are no longer convex. As a result, the mixed strategies can't be characterized as solutions to differential equations.

However, the problem that prevents convergence to efficiency seems to be caused by the local monopoly that firms and workers at the lower end of the distribution of investments enjoy. This local monopoly persists as the market grows large. The worst firm and worker simply have no incentive to invest. They drag the rest of the market down with them. This phenomena also exists when costs are different, so the same inefficiency of investment should be expected.

Another alternative is to imagine that workers and firms have private information about costs, the idea being that firms and workers aren't sure when they are the 'worst'. This is misleading however, since once the distribution of possible types is specified, the support has a lower bound, and the equilibrium investment of this type must be specified. This type has no incentive to invest when bidding strategies are monotonic, so the same kind of inefficiency emerges. One way to get some insight into the distribution of types approach to this problem is to look at Lemma9.3 in the appendix which purifies the mixed strategy equilibrium discussed through most of the paper. The purified type contingent strategies are monotonic when m and n are finite but converge in the limit to discontinuous step functions.

This purification is somewhat special in that the preferences of the different types are all the same. Yet is shows how the basic properties of the limiting result would remain the same in a type based model with monotonic bidding strategies.

Allowing transfer payments that depend on the ex post realized investments of the buyers and sellers may change the conclusions of this paper.[14] The effect of transfers depends on how they are introduced. Felli and Roberts (2000) allow firms to bid for workers once all investments are realized. This bidding is complicated since the wage that a firm wants to offer depends on the worker it is trying to attract. If the gain to the worker to a match with a firm is given entirely by the wage that it is paid, then they show that there will be persistent underinvestment as the market becomes large. This result is driven by the local monopoly that persists at the lower end of the market. The worst employed worker, for example, only attracts a wage high enough to make him indifferent between working and being unemployed. Any additional investment the worker makes will raise his payoff only by as much as it improves his utility when he is unemployed even if the gain to the firm of the investment is very large. So a holdup problem persists.

However this result depends on the fact that the only gain a worker enjoys from matching with a higher quality firm is the higher wage the firm pays. If the higher quality firm generates a higher payoff for the worker independent of the wage the firm pays, then the arguments made above kick in. The strategic complementarities introduced here are partly offset by the holdup problems and the overall effect of welfare is unclear.

A reasonable conjecture is that fully non-cooperative investments and transfers might lead to efficient investment if firms, for example, could commit themselves to wage schedules in which ex post payments to workers could depend on observable investment. One immediate complication associated with this is whether firms have the right incentives to offer the correct wages off equilibrium. Another complicating factor is that firms who offer arbitrary wage schedules don't obviously want to match ex post with the highest quality worker. So arguments based on assortative matching need to be carefully reconsidered in such a model.

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[14]For example, using a cooperative solution concept to determine transfers and matching ex post leads to efficient investment as in Cole, Mailath, and Postlewaite (2001b). To oversimplify the argument drastically, the core requires that all firms of the same type receive the same payoff, so the payoff of any potential partner for a worker is fixed (as long as there are other firms of the same type in the market. The benefits of any additional investment that the worker makes are then entirely realized by the worker in a core solution. This resolves the holdup problem and gives efficiency.

7. Conclusion

Efficient investment in the pre-marital investment game cannot typically be supported as a limit of Nash equilibria of the same game. Even though approximation of competitive allocation with Nash allocations in large finite games does not work perfectly, the limits do show how to understand the off equilibrium problem. The competitive problem seems to allow a profitable deviation by anyone who understands the nature of the matching process. The large finite games show how this behavior can be rationalized.

From the broader perspective of matching theory, the pre-marital investment game illustrates why it might be important to worry about the fact that the preferences of parties being matched might be endogenous. The worker firm matching problem is very special in the sense that workers and firms are assortatively matched ex post. What this really means is that every worker agrees what the best firm is, what the second best firm is, and so on. Similarly for firms. Whether the distorted investment incentives illustrated here persist when endogenous characteristics are multi-dimensional, and where parties disagree about who the best partner would be, is an important open question.

8. Proof of Proposition 4.2

Recall that $n = \tau m$ for some constant τ. In this section limits taken as n goes to infinity are equivalently limits as m goes to infinity. It should be understood that all random variables are drawn from the equilibrium distributions G_n and F_m when these aren't mentioned explicitly. Following the notational convention in the rest of the paper, let $\tilde{G}_{n:n'}$ and $\tilde{F}_{m:m'}$ denote the realized distribution functions when n' (or m' respectively) independent draws are made from the distribution G_n (or F_m). By the definition of 'order statistic', $\tilde{G}_{n:n'}(\tilde{h}_{n':i}) = \frac{i}{n'}$.

Now consider a firm who plays his or her bilateral Nash investment. Since the Nash investment is in the support of firms' equilibrium strategy for all m by Lemma 3.2, his or her payoff must coincide with firms' equilibrium payoff. Since this firm will have the lowest investment among firms with probability 1 in equilibrium by Lemma 3.1, this firm will match with the worker who has the $n - m + 1^{st}$ lowest investment. As many workers randomize with small but positive probability over investments above h^*, the expectation of this random variable must be bounded away from h^*even if the equilibrium distributions become degenerate in the limit.

Lemma 8.1. *Suppose* $\lambda < \frac{\tau-1}{\tau}$ *, then* $\lim_{m\to\infty} \mathbb{E}_{G_n} \tilde{h}_{n:n-m+1} > h^*$ *whenever this limit exists.*

Proof. $\lambda < \frac{\tau-1}{\tau}$ implies that $\lambda n < n - m$. Since all regular workers invest at least h^*, the $n-m+1^{st}$ order statistic of workers' investments can only be less than h^* if at least $n-m$ of all workers are poor workers. The probability with which this event occur is given by the binomial formula

$$P_n \equiv \sum_{k=0}^{m-1} \frac{(n-1)!}{(n-m+k)!(m-k-1)!} \lambda^{(n-m+k)} (1-\lambda)^{(m-k-1)}$$

This is the same as the probability with which the proportion $\frac{n-m-1}{n}$ or more of all other workers are poor workers. By standard theorems (for example (Shorak and Wellner 1986) Theorem 2, p 483), this probability goes to zero with n since $\lambda < \frac{\tau-1}{\tau}$. So $\lim_{m\to\infty} \mathbb{E}_{G_n} \tilde{h}_{n:n-m+1} \geq h^*$. Now suppose that contrary to the assertion in the Lemma,

$$\lim_{m\to\infty} \mathbb{E}_{G_n} \tilde{h}_{n:n-m+1} = h^*$$

Let $\eta_m(h)$ be the distribution function of the $n-m+1^{st}$ lowest investment of workers conditional on this $n-m+1^{st}$ lowest investment being made by a regular worker. The support of η_m has a lower bound equal to h^* so $\mathbb{E}_{G_n} \tilde{h}_{n:n-m+1} = (1-P_n) \int_{h^*}^{\overline{h}} \tilde{h} d\eta_m(h)$. If this converges to h^*, then for any $h' > h^*$ and any ϵ, there must be some m' such that $\eta_{m'}(h') > 1-\epsilon$. So if the contrary hypothesis is true, for any such m' the payoff to an investment h' is at least $u(h') + (1-P_n)(1-\epsilon)k^* > h^*$. On the other hand, a worker who invests h^* will rank behind all regular workers with probability 1 by Lemma 3.1. So the investment h^* yields payoff no higher than $u(h^*) + P_n \overline{k}$. So for some m large enough, $h' > h^*$ gives a strictly higher expected partner. Since h^* must be in the support of the equilibrium strategy by Lemma 3.2, this profitable deviation provides a contradiction. □

The next Lemma shows that G_n must converge to a degenerate strategy that has a flat segment to the right of h^*.

Lemma 8.2. *Let* $h^* < h' < \lim_{m\to\infty} \mathbb{E}_{G_n} \tilde{h}_{n:n-m+1}$. *Then along any sequence for which* G_n *converges weakly,*

$$\lim_{m\to\infty} G_n(h') = \lim_{m\to\infty} \frac{n-m-1}{n} = \frac{\tau-1}{\tau}$$

Proof. Suppose first that

$$\frac{\tau-1}{\tau} - \lim_{m\to\infty} G_n(h') = \epsilon > 0$$

By Lemma 9.2 (in the Appendix), $\Pr\left\{\left|\tilde{G}_{n:n}\left(h'\right) - G_n(h')\right| > \epsilon\right\} < \epsilon$ for large enough m. Then $\Pr\left\{\tilde{G}_{n:n-1}(h') > \frac{n-m}{n}\right\} < \epsilon$ provided m (equivalently $n-1$) is large enough. This makes the expected quality of a worker's partner when he invests h' less than or equal to

$$\Pr\left\{\tilde{G}_{n:n-1}(h') > \frac{n-m}{n}\right\}\overline{k} < \epsilon\overline{k}$$

Since ϵ can be taken to be arbitrarily small, and since $h' > h^*$, it must be that the payoff associated with the investment h' is less than $u(h^*)$ for m large enough. This means that h' lies outside the support of the equilibrium strategy for m large enough. This contradicts the assumption that $\mathbb{E}_{G_n}\tilde{h}_{n:n-m+1} > h'$.

On the other hand, if $\lim_{m\to\infty} G_m\left(h'\right) > \frac{\tau-1}{\tau}$, then using similar reasoning as above,

$$\Pr\left\{\tilde{G}_{n:n}(h') < \frac{\tau-1}{\tau}\right\} < \epsilon$$

for arbitrary ϵ provided m is large enough. This implies that

$$\mathbb{E}_{G_n}\tilde{h}_{n:n-m+1} < h'$$

for large enough m, a contradiction. □

Lemma 8.3. *Let h'' be any point that is in the support of G_m for infinitely many m and $\lim_{m\to\infty} G_m\left(h''\right) > \frac{\tau-1}{\tau}$. Let u^e_m be the* equilibrium *payoff of workers. Then $u^e_m \geq u(h^*)$ and $u\left(h''\right) + k^* \leq \lim_{m\to\infty} u^e_m = u(h^*)$.*

Proof. Again using Lemma 9.2, $\tilde{G}_{m:m}\left(h''\right)$ converges in probability to something that strictly exceeds $\frac{\tau-1}{\tau}$. So any worker who invests h'' will match with *some* firm with probability arbitrarily close to one when m is very large. Since no firm invests less than k^* the expected investment of the worker's partner in equilibrium must then be at least k^*. $u^e_m \geq u(h^*)$ since a worker who invests h^* will match with some firm because of the poor workers. The limit result follows because $\lambda < \frac{\tau-1}{\tau}$, and because a worker who invests h^* invests less than all regular workers with probability 1. □

Since the set of probability distributions on $\overline{H}$ is closed under the weak topology, we can assume that there is a sub-sequence along which $\lim_{m\to\infty} G_m\left(\cdot\right)$ has a weak limit which is itself a distribution function. Let G be any such weak limit. Let $h_0 \equiv \inf\left\{h_i : G\left(h_i\right) > \frac{\tau-1}{\tau}\right\}$. For each $h_i \in [h^*, h_0]$ define $\kappa\left(h_i\right)$ to be the solution to $u\left(h_i\right) + k_j =$

$u(h^*)$. The next Lemma is an immediate consequence of Lemma 8.2 and Lemma 8.3;

Lemma 8.4. *For any sequence of mixed strategies G_m converging weakly to G, there is an $h_0 \geq \{h_i : u(h_i) + k^* = u(h^*)\}$ such that G satisfies*

$$G(h') = \begin{cases} \lambda & \text{if } h' < h^* \\ \frac{\tau-1}{\tau} & \text{if } h^* \leq h' < h_0 \\ > \frac{\tau-1}{\tau} & \text{otherwise} \end{cases}$$

The next Lemma establishes some limit results for investments above and below the critical point h_0.

Lemma 8.5. *Suppose $h'' > h_0 > h'$. Let $\delta = G(h'') - \frac{\tau-1}{\tau} > 0$. Then*

$$\lim_{n\to\infty} \Pr\left\{\tilde{G}_{n:n-1}(h'') > \frac{n-m}{n} + \frac{\delta}{2}\right\} = 1$$

and

$$\lim_{n\to\infty} \Pr\left\{\tilde{G}_{n:n-1}(h') > \frac{n-m}{n} + \frac{\delta}{2}|\tilde{h}_{n:n-m} < h'\right\} = 0$$

Proof. As $h'' > h_0$, $G(h'') > \frac{\tau-1}{\tau}$. Since G is non-decreasing, it is continuous almost everywhere, so h'' can be chosen so that $G(h'')$ is continuous. Then by weak convergence,

$$\lim_{m\to\infty} G_n(h'') - \frac{\tau-1}{\tau} = G(h'') - \frac{\tau-1}{\tau} = \delta = \lim_{m\to\infty} G(h'') - \frac{n-m}{m}$$

In particular, this implies that $G_n(h'') > \frac{n-m}{m} + \frac{3\delta}{4}$ for infinitely many n. For every n for which this last inequality is satisfied,

$$\Pr\left\{\left|\tilde{G}_{n:n-1}(h'') - G_n(h'')\right| > \frac{\delta}{4}\right\} \geq$$

$$\Pr\left\{\tilde{G}_{n:n-1}(h'') \leq G_n(h'') - \frac{\delta}{4}\right\} \geq$$

$$\Pr\left\{\tilde{G}_{n:n-1}(h'') \leq \frac{n-m}{m} + \frac{\delta}{2}\right\}$$

Since the probability expression in the first line converges to zero with n by Lemma 9.2, the last one must also.

When $h' < h_0$, consider any event in which $\tilde{h}_{n-1:n-m} < h'$. The remaining m investments are independently drawn from the conditional distribution

$$G_n(h_i|\tilde{h}_{n-1:n-m}) = \begin{cases} \frac{G_n(h_i)-G_n(\tilde{h}_{n-1:n-m})}{1-G_n(\tilde{h}_{n-1:n-m})} & \text{if } h' > \tilde{h}_{n-1:n-m} > 0 \\ G_n(h_i) & \text{if } \tilde{h}_{n-1:n-m} = 0 \end{cases}$$

Taking expectations gives

$$G_n(h_i|h') \equiv$$

$$\Pr\left\{\tilde{h}_{n-1:n-m} = 0\right\} G_n(h_i) + \int_{h^*}^{h'} \frac{G_n(h_i) - G_n(\tilde{h}_{n-1:n-m})}{1 - G_n(\tilde{h}_{n-1:n-m})} d\gamma_n \leq$$

$$\Pr\left\{\tilde{h}_{n-1:n-m} = 0\right\} G_n(h_i) + \frac{G_n(h_i) - G_n(h^*)}{1 - G_n(h^*)}$$

where γ_n is the (conditional) distribution of the random variable $\tilde{h}_{n-1:n-m}$ on the interval $(h^*, h']$. Since the last expression converges to zero for each $h' < h_i < h_0$, $G_n(h_i|h')$ converges to zero for h_i in the same interval.

Then for any weak limit $G(\cdot|h')$, standard properties of weak convergence give,

$$\limsup G_n\left(h'|h'\right) \leq G\left(h_0 - \frac{h_0 - h'}{2}|h'\right) = 0$$

Emulating the previous argument then gives

$$\lim_{n\to\infty} \Pr\left\{\tilde{G}_{n:n-1}(h') > \frac{\delta}{2}|\tilde{h}_{n-1:n-m} \leq h'\right\} = 0.$$

□

The next Lemma is the critical one, that gives the bound on the expected quality of the partner of a firm who invests k^*. This bound limits the location of firms equilibrium indifference curve because of the fact that k^* is always in the support of the firms' equilibrium strategy.

Lemma 8.6. *For any sequence of equilibrium mixed strategies G_n converging weakly to G,*

$$\lim_{m\to\infty} \mathbb{E}\tilde{h}_{n:n-m+1} \leq \frac{1}{\kappa\left(h_0\right)} \int_{h^*}^{h_0} \tilde{h} d\kappa\left(\tilde{h}\right)$$

where $\kappa(h_i) = \{k_j : u(h_i) + k_j = u(h^*)\}$ and h_0 is the value described in Lemma 8.4.

Proof. $\tilde{h}_{n-1:n-m}$ is the $n-m^{th}$ lowest investment among the $n-1$ investments of the 'other' workers when each of the other $n-1$ workers' investments are independently drawn from G_m. A worker who invests h_i will match with some firm if $\tilde{h}_{n-1:n-m} < h_i$ (by Lemma 3.1 $\tilde{h}_{n-1:n-m} = h_i$ occurs with zero probability). If $\tilde{h}_{n-1:n-m} > h_i$ the worker will not be matched with any firm. Since the event $\tilde{h}_{n-1:n-m} < h_i$ must occur with probability strictly bounded above zero for all m and all

$h_i > h^*$(in order for an investment $h_i > h^*$ to produce expected payoff u_m^e for all m) it must be the case that the conditional expectation $\mathbb{E}_{G_n,F_m}\left\{\tilde{k}|h_i;\tilde{h}_{n-1:n-m} \leq h_i\right\}$ exists for each $h^* < h_i < h_0$, (here the expectation means the expected quality of the worker's partner conditional on matching with investment h_i). So by Lemma 8.5 and the fact that workers are matched assortatively with firms, when n is large enough

$$\mathbb{E}_{G_n,F_m}\{\tilde{k}|h'\} = \Pr\left\{\tilde{h}_{n-1:n-m} \leq h'\right\}\mathbb{E}_{G_n,F_m}\left\{\tilde{k}|h';\tilde{h}_{n-1:n-m} \leq h'\right\}$$
$$\leq \Pr\left\{\tilde{h}_{n-1:n-m} \leq h'\right\}\mathbb{E}_{G_n,F_m}\tilde{k}_{m:\frac{\delta}{2}m}$$
$$\leq \Pr\left\{\tilde{h}_{n-1:n-m} \leq h'\right\}\mathbb{E}_{G_n,F_m}\left\{\tilde{k}|h''\right\}$$

In these expressions $\frac{\delta}{2}m$ means the largest integer that is less than or equal to $\frac{\delta}{2}m$.

Then

$$\kappa(h') = \lim_{m\to\infty}\mathbb{E}_{G_n,F_m}\{\tilde{k}|h'\} =$$
$$\lim_{m\to\infty}\Pr\left\{\tilde{h}_{n-1:n-m} \leq h'\right\}\mathbb{E}_{G_n,F_m}\left\{\tilde{k}|h';\tilde{h}_{n-1:n-m} \leq h'\right\} \leq$$
$$\lim_{m\to\infty}\Pr\left\{\tilde{h}_{n-1:n-m} \leq h'\right\}\mathbb{E}_{G_n,F_m}\left\{\tilde{k}|h''\right\} =$$
$$\lim_{m\to\infty}\Pr\left\{\tilde{h}_{n:n-m+1} \leq h'\right\}\mathbb{E}_{G_n,F_m}\left\{\tilde{k}|h''\right\} =$$
$$\kappa(h'')\cdot\lim_{m\to\infty}\Pr\left\{\tilde{h}_{n:n-m+1} \leq h'\right\}$$

The second to last inequality follows from the fact that the random variables $\tilde{h}_{n:n-m+1}$ and $\tilde{h}_{n-1:n-m}$ have almost the same distribution when n is large. This is proved formally in the appendix in Lemma 9.3. Recall that $\kappa(h_i)$ is such that $u(h_i)+\kappa(h_i)$ is the limit of the expected payoff to a worker who invests h_i against the equilibrium mixed strategies of the other players. Since this limit result is true for all $h'' > h_0$, and the function $\kappa(\cdot)$ is continuous we have

$$\lim_{m\to\infty}\Pr\left\{\tilde{h}_{n:n-m+1} \leq h'\right\} \geq \frac{\kappa(h')}{\kappa(h'')}$$

for each $h^* \leq h' \leq h_0$. So every weak limit of the probability distribution function $\Pr\left\{\tilde{h}_{n:n-m+1} \leq h'\right\}$ is stochastically dominated by the probability distribution $\frac{\kappa(\cdot)}{\kappa(h_0)}$. The statement in the Theorem follows from this. □

Finally, we can complete the proof of Proposition 4.2:

Proof. Let $h'' > h_0$, where h_0 is the investment level described in Lemma 8.4 where the limit investment strategy has an atom. Let F be some weak limit of the sequence F_m and suppose that $k_0 = \inf\{k : F(k) > 0\}$. Since F is continuous (constant) for all $k_j < k_0$, $\Pr\left\{\tilde{F}_{m:m}(k_j) > \epsilon\right\}$ converges to zero for every ϵ by Lemma 9.2. Let $G(h'') = \frac{\tau-1}{\tau} + \delta$. Then by Lemma 8.5, $\Pr\left\{\tilde{G}_{n:n-1}(h'') > \frac{n-m}{m} + \frac{\delta}{2}\right\}$ converges to one. So for any $k_j < k_0$, $\Pr\left\{\tilde{F}_{m:m}(k_j) \geq \tilde{G}_{n:n-1}(h'') - \frac{n-m}{n}\right\}$ converges to zero. So any worker who invests $h'' > h_0$ will match with high probability as m gets large with a firm whose investment is at least k_0. In other words, $\lim_{m\to\infty} u(h'') + \mathbb{E}_{G_n,F_m}\left\{\tilde{k}|h''\right\} \geq u(h'') + k_0$ for all $h'' > h_0$. The function u is continuous, $u(h_i) + \mathbb{E}_{G_n,F_m}\left\{\tilde{k}|h_i\right\}$ is constant for all h_i in the support of workers' equilibrium strategy, and h_0 is in this support for infinitely many m by the definition of G and Lemma 3.2. So $\lim_{m\to\infty} u(h_0) + \mathbb{E}_{G_n,F_m}\tilde{k}|h_0 \geq u(h_0) + k_0$.

If the inequality is strict, then there is an investment $h_i < h_0$ such that $\lim_{m\to\infty} u(h_i) + \mathbb{E}_{G_n,F_m}\tilde{k}|h_i > u(h_0) + k_0$, or $\mathbb{E}_{G_n,F_m}\tilde{k}|h_i > k_0$. Then reasoning as above, for any $k_j > k_0$,

$$\Pr\left\{\tilde{G}_{n:n-1}(h_i) - \frac{n-m}{n} > \tilde{F}_{m:m}(k_j)\right\}$$

converges to zero, so a worker who invests $h_i < h_0$ will match with high probability with a firm whose investment is no higher than k_0, which gives $\mathbb{E}_{G_n,F_m}\tilde{k}|h_i \leq k_0$, a contradiction. So

$$\lim_{m\to\infty} u(h_0) + \mathbb{E}_{G_n,F_m}\tilde{k}|h_0 = u(h_0) + k_0.$$

Exactly the same reasoning shows that

$$\lim_{m\to\infty} v(k_0) + \mathbb{E}_{G_n,F_m}\left\{\tilde{h}|k_0\right\} = v(k_0) + h_0.$$

The fact that $h_0 > h^e$ can be seen with the help of Figure 2. Suppose to the contrary that (h_0, k_0) lies somewhere below and to the left of the tangency marked in that figure. By the last result, firms attain the payoff $v(k_0) + h_0$ in the limit. The function $\frac{1}{\kappa(h_0)}\int_{h^*}^{h_0} \tilde{h} d\kappa\left(\tilde{h}\right)$ is strictly increasing in h_0, so by Condition 4.1 Lemma 8.6, the indifference curve that firms attain in the limit by investing k^* must lie strictly to the left of the one through (h_0, k_0). This contradicts the fact that both k^* and k_0 must yield the same limit payoff to firms.

So in what follows, it is assumed that $h_0 > h^e$. Now suppose that $G(h_0) < 1$. Since G is right continuous, there is h^1 such that $h_0 <$

$h^1 < \inf \{h_i : G(h_i) = 1\}$. Again, since G is non-decreasing, the set of its discontinuity points is countable. So it is possible to choose h^1 so that G is continuous at h^1. Since $G(h^1) < 1$, $G_m(h^1) < 1$ for large m, and in particular h^1 is in the support of G_m for infinitely many m.

Now $u^e_m \downarrow u(h^*)$ and for each $m > m^*$, h^1 must yield a worker the equilibrium payoff. Hence a worker who plays h^1 expects a partner with quality k^1 such that $u(h^1) + k^1 = u^e_m$. Since $h_0 > h^e$ and a firm who invests k_0 has limit payoff equal to $v(k_0) + h_0$ as shown above, it must be that for large enough m, $v^e_m > v(k_1) + h_1$. Then by Lemma 9.1, h_1 cannot be part of the support of any equilibrium strategy. This contradiction proves the result. □

9. Appendix - Proofs of Other Theorems and Lemmas

9.1. **Proof of Lemma 3.3:** To prove the lemma, we prove a result that will be useful later.

Lemma 9.1. *Let (h', k') be a pair such that workers' equilibrium payoff is equal to $u(h') + k'$, firms' equilibrium payoff is strictly greater than $v(k') + h'$, and $-v'(k') > -u'(h')$. Then neither h'nor k' can be part of the support of workers' or firms' equilibrium strategies.*

Proof. Let $\kappa(\tilde{h}) = \left\{k_j : u(\tilde{h}) + k_j = u^e\right\}$ be the graph of an unmatched workers' equilibrium indifference curve (where u^e is workers' equilibrium payoff). Define the mapping $\gamma(\tilde{h}) \mapsto \left\{h_i : v(\kappa(\tilde{h})) + h = v^e\right\}$. For any pair $h_1 > h_0$

$$\gamma(h_1) - \gamma(h_0) =$$

$$\begin{gathered} \{h_i : v(\kappa(h_1)) + h_i = v^e\} - \{h_i : v(\kappa(h_0)) + h_i = v^e\} > \\ \{h_i : v(\kappa(h_0) + \kappa'(h_0)(h_1 - h_0)) + h_i = v^e\} - \{h_i : v(\kappa(h_0)) + h_i = v^e\} = \\ v(\kappa(h_0)) - v(\kappa(h_0) + \kappa'(h_0)(h_1 - h_0)) > \\ -v'(\kappa(h_0))\kappa'(h_0)(h_1 - h_0) > \\ h_1 - h_0 \end{gathered} \tag{4}$$

The first inequality follows because κ is strictly convex. The equality follows because firm payoffs are linear in h_i. The next inequality follows because v is strictly concave and decreasing when $h_i > h^*$. The final inequality follows from the fact that $\kappa'(h_0) = -\frac{1}{u'(h_0)}$ and the assumption that $-v'(\kappa(h_0)) > -u'(h_0)$.

If h' is in the support of workers' equilibrium strategy, then $k' = \kappa(h')$ must be in the support of firms' equilibrium strategy, otherwise by Lemma 3.2, workers could never expect to match with a firm with such a high investment. Since firms achieve a payoff higher than $v(\kappa(h'))+h'$ in equilibrium, investments strictly larger than h' must lie in the support of workers' equilibrium strategy - specifically, investments at least as large as $\gamma(h') > h'$.

If workers use investments as large as $\gamma(h')$ in equilibrium, then investments at least as large as $\kappa(\gamma(h'))$ must be in the support of firms' equilibrium strategy, requiring investments as large as $\gamma(\gamma(h'))$, to be in workers' strategy, and so on. Reasoning this way, if h' is in the support of workers' equilibrium strategy, then investments as large as $\gamma^n(h')$ also lie in the support of their equilibrium strategy for arbitrarily large n. However, by (4), the series $\gamma^n(h')$ diverges, and must exceed the upper bound on feasible investments for some n. This contradiction gives the result. □

Lemma. *(Restatement of Lemma 3.3)If symmetric equilibrium strategies G and F exist, then their supports are bounded above by $\overline{h}$ and $\overline{k}$.*

Proof. Choose some investment $h' > \overline{h}$. If this investment is in the support of workers' equilibrium strategy, then the worker expects to match with a partner whose investment is $k' : u(h') + k' = u^e$. It is straightforward from the definition of $(\overline{h}, \overline{k})$ that $v(k') + h' < v^e$, and $-v'(k') > -u'(h')$. The result then follows by Lemma 9.1. □

9.2. **Proof of Lemma 3.4 :** Restatement of Lemma 3.4: Let F be a distribution with support contained in the interval $[k^*, \overline{k}]$. Then there is a unique symmetric best reply G for workers with a convex support contained in $\overline{H} \equiv \{h_i \in H : \overline{h} \geq h_i \geq h^*\}$. The point h^* is contained in the support of this symmetric best reply.

Proof. (3) is an ordinary differential equation on the interval $[h^*, \overline{h}]$ with initial condition $G(h^*) = \lambda$. The equation depends on a vector of m parameters $\left\{\mathbb{E}_F \tilde{k}_{m:1}, \ldots, \mathbb{E}_F \tilde{k}_{m:m}\right\}$. By standard theorems (e.g Kreider, Kuller, and Ostberg (1968)) for differential equations, the equation will have a unique solution that varies (sup norm) continuously in these parameters if it is Lipshitzian on a region $\left\{(h_i, G) : h_i \leq \overline{h}; G \leq 1\right\}$, in other words if the derivative of the right hand side of (3) bounded above in h and G in that region. Write (3) as

$$G'(h_i) = \frac{-u'(h_i)}{\Pi'(G(h_i)) \cdot \mathbb{E}_F \tilde{k}_{m:\cdot}}$$

To give $\mathbb{E}_F \tilde{k}_{m:\cdot}$ the right dimension, suppose that for the purposes of this argument, it is augmented by adding 0 to the first $n-m$ positions in the vector. $\Pi'(G)$ is the vector of derivatives of the binomial probabilities

$$\binom{n-1}{t} G^t (1-G)^{n-1-t}$$

Observe that $\Pi(G) \cdot e \equiv 1$ for all G (e is an n vector consisting entirely of 1's) so $\Pi'(G) \cdot e = 0$, and that $\Pi'_t(G) < 0$ for each $t < G(n-1)$; $\Pi'_t(G) = 0$ if $t = G(n-1)$ and $\Pi'_t(G) > 0$ otherwise. These are standard properties of the binomial distribution function. Also note that the second derivative $\Pi''(G)$ will involve a finite sum of finite terms, so will be finite. Thus (3) will be Lipshitzian if $\Pi'(G) \cdot \mathbb{E}_F \tilde{k}_{m:\cdot} > 0$ (for all G in $[0,1]$).

Formally for and $G \in [0,1]$

$$\Pi'(G) \cdot \mathbb{E}_F \tilde{k}_{m:\cdot} \geq$$

$$\sum_{t=0}^{(n-m-1)} \binom{n-1}{t} G^{t-1}(1-G)^{n-2-t} \cdot \mathbb{E}_F \tilde{k}_{m:t-G(n-1)} +$$
$$\sum_{t=n-m}^{G(n-1)} \left\{ \binom{n-1}{t} G^{t-1}(1-G)^{n-2-t} \cdot \right.$$
$$\left. \cdot (t - G(n-1)) \mathbb{E}_F \tilde{k}_{m:G(n-1)-(n-m-1)} \right\} +$$
$$\sum_{t=G(n-1)+1}^{n-1} \left\{ \binom{n-1}{t} G^{t-1}(1-G)^{n-2-t} \cdot \right.$$
$$\left. \cdot (t - G(n-1)) \mathbb{E}_F \tilde{k}_{m:G(n-1)-(n-m-1)+1} \right\}$$

Write

$$\eta_1 = \sum_{t=0}^{(n-m-1)} \binom{n-1}{t} G^{t-1}(1-G)^{n-2-t}(t - G(n-1)),$$
$$\eta_2 = \sum_{t=n-m}^{G(n-1)} \binom{n-1}{t} G^{t-1}(1-G)^{n-2-t}(t - G(n-1))$$

and observe that

$$\eta_1 + \eta_2 =$$
$$\sum_{t=G(n-1)+1}^{n-1} \binom{n-1}{t} G^{t-1}(1-G)^{n-2-t}(t - G(n-1))$$

This gives

$$\Pi'(G) \cdot \mathbb{E}_F \tilde{k}_{m:\cdot} \geq$$

$$-\eta_1 \cdot 0 - \mathbb{E}_F \tilde{k}_{m:G(n-1)-(n-m-1)} \eta_2 +$$

$$
\begin{aligned}
&+\mathbb{E}_F \tilde{k}_{m:G(n-1)-(n-m-1)+1} (\eta_1 + \eta_2) \\
&\geq \eta_1 \left(\mathbb{E}_F \tilde{k}_{m:G(n-1)-(n-m-1)+1} \right) \\
&+\eta_2 \left(\mathbb{E}_F \tilde{k}_{m:G(n-1)-(n-m-1)+1} - \mathbb{E}_F \tilde{k}_{m:G(n-1)-(n-m-1)} \right) \\
&\geq \eta_1 k^* + \eta_2 \left(\mathbb{E}_F \tilde{k}_{m:G(n-1)-(n-m-1)+1} - \mathbb{E}_F \tilde{k}_{m:G(n-1)-(n-m-1)} \right)
\end{aligned}
$$

Since $k^* > 0$, $\Pi'(G) \cdot \mathbb{E}_F \tilde{k}_{m:\cdot} > 0$ provided η_1 and η_2 are both strictly positive, which is true whenever $G > 0$. Since the initial condition has $G(h^*) = \lambda > 0$, this gives the result.

In when proving this condition in the solution to F, the initial condition is that $F(k^*) = 0$, so we verify that the derivative is also bounded when $G = 0$. The argument is almost identical (except for changes in indices) for firms.

Rewrite the denominator of (3) as

$$
\frac{\sum_{t=0}^{n-1} \binom{n-1}{t} G^t (1-G)^{n-1-t} (t - G(n-1)) \mathbb{E}_F \tilde{k}_{m:t-(n-m-1)}}{G(1-G)}
$$

Both numerator and denominator go to zero with G. By L'Hopitals rule, the limit is given by taking the ratio of the limit of the derivatives of these expression, which are both strictly positive since $\mathbb{E}_F \tilde{k}_{m:t-(n-m-1)}$ is not a constant sequence.[15] Hence this derivative is strictly positive.

Let B be the Lipschitz bound on the right hand side of (3). By standard existence theorems for differential equations (e.g. Kreider, Kuller, and Ostberg (1968) Theorem 9.6 p.378) the differential equation (3) with initial value (h^*, λ) has a unique solution on some the interval $[h^*, h_1]$ where $B(h_1 - h^*) = 1 - G(h^*)$. If $G(h_1) < 1$, then apply the theorem again to (3) with initial value $(h_1, G(h_1))$. Since the Lipschitz bound B doesn't change on this new region, the resulting solution extends uniquely on a non-degenerate interval $[h^*, h_2]$ where $B(h_2 - h_1) = 1 - G(h_1)$. If $G(h_2) < 1$, then repeat this procedure until either $G(h_l) = 1$, or $h_l > \overline{h}$. To complete the argument, observe that if $h_l > \overline{h}$ (meaning that $G(h_l) < 1$), then a worker who invests h_l will achieve the same expected payoff as a worker who invests h^*. Since a worker who invests h^* will match with some firm if enough workers are poor workers, this payoff must be at least $u(h^*)$. To accomplish this, the expected quality of the worker's partner must exceed $\overline{k}$, since by definition, $u(\overline{h}) + \overline{k} = u(h^*)$. This is impossible since $\mathbb{E}_F \tilde{k}_{m:t} \leq \overline{k}$ for all

[15] Even if all firms make the same investment, it must be at least k^*. Each term $\mathbb{E}_F \tilde{k}_{m:t}$ with $t < m - n$ on the other hand must be zero, so the entire vector cannot be constant. It is for this reason that we need poor workers to ensure the non-constancy of this same vector in the verification of the firms' Lipschitz condition.

t. This establishes that the solution to the differential equation must attain the value $G(h_l) = 1$ for some finite l such that $h_l < \overline{h}$. This completes the proof. □

Almost the same argument applies for firms, though all firms will match in any equilibrium of this kind so the term involving η_1 in the proof that $\Pi'(G)\,\mathbb{E}_F\tilde{k}_{m:\cdot}$ will not exist. The poor worker assumption ensures that the term $\mathbb{E}_G\tilde{h}_{m:F(k)(n-1)+1} - \mathbb{E}_G\tilde{h}_{m:F(k)(n-1)}$ which multiplies η_2 will be strictly positive.

9.3. Proof of Lemmas Used in the Proof of Proposition 4.2.

Lemma 9.2. *Let G_n and F_m be sequences of equilibrium strategies that converge weakly to G and F. Then for any $\delta > 0$*

$$\Pr\left\{\sup_{\tilde{h}} |\tilde{G}_{n:n}(\tilde{h}) - G_n(\tilde{h})| > \delta\right\}$$

converges to zero with n. Furthermore, if h_0 is a point at which $G(h_0)$ is continuous, then

$$\Pr\left\{|\tilde{G}_{n:n}(h_0) - G(h_0)| > \delta\right\}$$

also converges to zero.

Proof. In Shorak and Wellner (1986) Theorem 1, p. 106, it is shown that random sequences of empirical distribution functions like $\left\{\tilde{G}_{n:n}(\cdot), \tilde{F}_{m:m}(\cdot)\right\}$ satisfy the following condition, $\left|\left|\tilde{G}_{n:n}(\cdot) - G_n(\cdot)\right|\right|$ (or $\left|\left|\tilde{F}_{m:m}(\cdot) - F_m(\cdot)\right|\right|$) converges almost surely to zero with m, where $||\cdot||$ is the sup norm for functions on $\overline{H}$. Since uniform convergence implies convergence in probability, the statement in the Lemma is an immediate consequence. □

Lemma 9.3. *For all $h_i \in [h^*, \overline{h}]$,*

$$\lim_{m\to\infty} \Pr\left\{\tilde{h}_{n:n-m+1} \le h_i\right\} = \lim_{m\to\infty} \Pr\left\{\tilde{h}_{n-1:n-m} \le h_i\right\}$$

Proof. The difference between the two is

$$\Pr\left\{\tilde{h}_{n-1:n-m} \le h_i < \tilde{h}_{n:n-m+1}\right\}$$

Observe that according to (3), $G'_m(h_i)$ exists and is strictly positive when $h_i \ge h^*$. As a consequence, we can *purify* the mixed strategy G_m in the following way - there is a strictly monotonic function $\rho_m : [0,1] \to \overline{H}$ such that for all h_i, $G_m(h_i) = \Pr_{U[0,1]}\{\tau : \rho_m(\tau) \le h_i\}$, where $\Pr_{U[0,1]}$ means that τ is drawn using a uniform distribution on $[0,1]$. Let $\tilde{\tau}_{n-1:n-m}$ be the $n-m^{th}$ lowest 'type' in $[0,1]$ when $n-1$ types are independently drawn from $[0,1]$ using the uniform distribution. Implicit in this notation is the fact that the distribution is the uniform distribution on $[0,1]$. Similarly let $\tilde{\tau}_{n:n-m+1}$ be the $n-m+1^{st}$ lowest type in $[0,1]$ when n types are drawn independently from $[0,1]$ using the uniform distribution. Then $\Pr\left\{\tilde{h}_{n-1:n-m} \leq h_i < \tilde{h}_{n:n-m+1}\right\} = \Pr\left\{\tilde{\tau}_{n-1:n-m} \leq \rho_m^{-1}(h_i) < \tilde{\tau}_{n:n-m+1}\right\}$. Since $\Pr\left\{\tilde{\tau}_{n-1:n-m} \leq \tau < \tilde{\tau}_{n:n-m+1}\right\}$ converges uniformly to zero on $[0,1]$ for the uniform distribution, the result is proved. □

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