EXISTENCE OF EQUILIBRIUM IN THE PRE-MARITAL INVESTMENT GAME

MICHAEL PETERS
DEPARTMENT OF ECONOMICS
UNIVERSITY OF BRITISH COLUMBIA

ABSTRACT. This paper proves existence in a fairly general version of the pre-marital investment game. The game has discontinuous payoffs, so the method of Reny (1999) is used. The key assumptions are that the matching process that occurs once investments have been made is assortative and resolves ties efficiently.

The pre-marital investment game is studied in Peters and Siow (2002) and Peters (2004). These papers consider special separable environments. The current note provides an existence theorem for a much more general version of the game. Preferences are continuous and monotonic in partner's characteristic, but otherwise arbitrary. The primary assumptions are on the matching process that occurs after investments have been observed. It is required to be assortative, and to resolve ties in an efficient manner.

1. FUNDAMENTALS

For ease of discussion this paper simply adopts the worker-firm version of the problem. The market consists of a finite number m of 'firms' and n of 'workers'. We use the notation M and N to refer to the sets of firms and workers respectively. It is assumed throughout that the measure of the set N is larger than M. Each firm has a characteristic x drawn from a closed connected interval $X \subset \mathbb{R}^+$. Each worker has a characteristic y which is again contained in a closed connected interval $Y \subset \mathbb{R}^+$. Firms and workers make investments $k \in K \subset \mathbb{R}^+$ and $h \in H \subset \mathbb{R}^+$ in physical capital and human capital respectively, where both K and H are assumed compact connected intervals.

Payoffs for firms and workers depend on their own type, on their own investment, and on the investment level of their eventual partner. The payoff of a firm is given by $v : K \times H \times X \to \mathbb{R}$, and that of a worker by $u : H \times K \times Y \to \mathbb{R}$. It is assumed that for each (x, h) the function $v(\cdot, h, x)$ is continuous, that $v(k, \cdot, x)$ is strictly increasing in h. Make

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the same assumptions for workers. A worker who is unmatched is assumed to receive a payoff $u^0(h,y) \leq u(h,0,y)$.

The matching process is given exogenously. A *matching* is an embedding $\pi : M \to N$. For convenience, write this embedding in its characteristic function form, i.e., $\pi_{ij} = 1$ if firm i is mapped into index j, and $\pi_{ij} = 0$ otherwise. Interpret $\pi_{ij} = 1$ to mean that firm i is matched with worker j. The set of workers who are not assigned partners by π is given by $J_\pi^- = \{j \in N : \pi_{ij} = 0 \forall i = 1, \ldots m\}$. Let Π be the set of matchings and $\triangle(\Pi)$ the set of probability distributions over the set of matchings. A *matching function* is a map $q : K^M \times H^N \to \triangle(\Pi)$. If $q_\pi(k,h)$ is the probability that matching π is implemented given investments (k,h), then $\sum_\pi q_\pi(k,h)\pi_{ij}$ is the probability with which firm i matches with worker j.

The matching function must satisfy two *regularity conditions*:

(1) (**Assortative Matching)** Let π be a matching such that $q_\pi(k,h) > 0$ for some array (k,h). For any pair of firms i and i', suppose that $\pi_{i'j'} > 0$, $\pi_{ij} > 0$ and $h_{j'} > h_j$. Then $k_i' \geq k_i$. (Similarly for workers).

(2) (**Ties Resolved Efficiently)** Let π be any matching such that $q_\pi(k,h) > 0$ for some array (k,h). Then there is no other *assortative* matching ρ such that

$$\sum_{i=1}^m \sum_{j=1}^n \pi_{ij}\left[v(k_i,h_j,x_i) + u(h_j,k_i,y_j)\right] + \sum_{j \in J_\pi^-} u(h_j,0,y_j) <$$

$$\sum_{i=1}^m \sum_{j=1}^n \rho_{ij}\left[v(k_i,h_j,x_i) + u(h_j,k_i,y_j)\right] + \sum_{j \in J_\rho^-} u(h_j,0,y_j)$$

The efficient matching condition says that ties are resolved in a way that maximizes the sum of the surplus generated by matching.

Payoff functions are given by

$$\sum_\pi q_\pi(k_i,k_{-i},h) \sum_{j=1}^n \pi_{ij} v(k_i,h_j,x_i) \tag{1.1}$$

for firm i, and

$$\sum_\pi q_\pi(k,h_j,h_{-j}) \sum_{i=1}^m \pi_{ij} u(h_j,k_i,y_j) \tag{1.2}$$

for worker j. An equilibrium for the pre-marital investment game is a Nash equilibrium for the normal form game defined by payoffs (1.1) and (1.2).

The simultaneous pre-marital investment game typically won't have pure strategy equilibria.[1] The payoff functions are discontinuous. For example, if two firms have the same investment, then one of them can strictly improve the expected quality of his or her partner by investing just slightly more than some other firm. The discontinuities in the game mean that standard existence theorems won't apply. Furthermore, the game has an unusual structure. Firms, for example, compete against other firms for partners. This is wholly standard, like a Bertrand pricing game, or an auction. Unlike standard games, the gains to winning the competition cannot be specified exogenously - they depend on the investment decisions of workers.

Theorem 1.1. *If the matching function satisfies the matching regularity conditions, then the investment game described by (1.1) and (1.2) has at least one mixed strategy Nash equilibrium.*

Proof. To verify the existence of a mixed strategy equilibrium, we take the approach of Reny (Reny 1999) and show that the mixed extension of the investment game is *reciprocally upper semi-continuous* and payoff secure. Observe first that the strategy spaces are assumed to be compact connected intervals in $\mathbb{R}^+$. So the investment game is a compact game with metric strategy spaces.

Let $\triangle_i$ be the compact set of regular countably additive probability measures on (the Borel sets of) the set $i = K, H$. The mixed extension of the first stage game is the game in which the firms' and workers' strategy spaces are given respectively by $\triangle_F$ and $\triangle_W$.

Let $\mu = \{\mu_1, \ldots \mu_m, \mu_{m+1}, \ldots \mu_{m+n}\}$ be the vector describing the mixed strategies of the firms, then the workers. The arguments that follow are completely symmetric for workers and firms, so focus on firms' payoffs. For any single firm, write the mixed strategies of the other firms and workers as μ_{-i}. The payoff of firm i in the mixed extension of the pre-marital investment game can be written as

$$V_i\left(k_i, \mu_{-i}, x_i\right) \equiv$$
$$\int \cdots \int \sum_{\pi} q_\pi\left(k, k_{-i}, h\right) \sum_{j=1}^{n} \pi_{ij} v\left(k_i, h_j, x_i\right) d\mu_{-i}$$

Define

$$\tilde{V}_i\left(\mu\right) \equiv \int V_i\left(k, \mu_{-i}, x_i\right) d\mu_i$$

A similar expression $\tilde{U}_j\left(\mu\right)$ can be written for each worker. (Note that j's type y_j is common knowledge in this game).

[1]See the example in (Peters and Siow 2002) which illustrates why.

Given vectors $k = (k_1, \ldots k_n)$ and $h = (h_1, \ldots h_m)$, the sum of the payoff functions is

$$S(k_1, \ldots h_n) =$$

$$\sum_{\pi} q_{\pi}(k,h) \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \pi_{ij} \left[v(k_i, h_j, x_i) + u(h_j, k_i, y_j) \right] \right.$$

$$\left. + \sum_{j \in J_{\pi}^{-}} u(y_j, h_j, 0) \right\}$$

where the matching function depends on the entire vector of investments. We will show that this sum is upper-semi-continuous in investments.

Suppose upper semi-continuity fails at (h,k). Then there is an $\varepsilon > 0$ and a sequence $\{h^{\tau}, k^{\tau}\} \to (h,k)$ such that $\lim_{(h^{\tau},k^{\tau}) \to (h,k)} S(h^{\tau}, k^{\tau}) > S(h,k) + \varepsilon$. Since the number of firms and workers are both finite, the number of ways to rank them is also finite. Hence we can choose a sub-sequence $\{k^n, h^n\}$ such that all traders' ranks in the distribution of investments remain unchanged along the sequence. In particular, do this in such a way that any strict ranking is preserved everywhere along the sequence. Then there is a single matching π^* that is efficient (among the set of all assortative matchings) for each element (h^n, k^n) of the sequence. By the definition of efficiency, the sum of the payoffs associated with any efficient match is the same. So

$$\lim_{(h^{\tau},k^{\tau}) \to (h,k)} \sum_{\pi} q_{\pi}(k^{\tau}, k_{-i}^{\tau}, h^{\tau}) \left\{ \sum_{i'=1}^{m} \sum_{j=1}^{n} \pi_{i'j} \left[v\left(k_{i'}^{\tau}, h_j^{\tau}, x_{i'}\right) + u\left(h_j^{\tau}, k_{i'}^{\tau}, y_j\right) \right] \right.$$

$$\left. + \sum_{j \in J_{\pi}^{-}} u\left(h_j^{\tau}, 0, y_j\right) \right\} =$$

$$\lim_{(h^{\tau},k^{\tau}) \to (h,k)} \left\{ \sum_{i'=1}^{m} \sum_{j=1}^{n} \pi_{i'j}^{*} \left[v\left(k_{i'}^{\tau}, h_j^{\tau}, x_{i'}\right) + u\left(h_j^{\tau}, k_{i'}^{\tau}, y_j\right) \right] \right.$$

$$\left. + \sum_{j \in J_{\pi}^{-}} u\left(h_j^{\tau}, 0, y_j\right) \right\} =$$

$$\left\{ \sum_{i'=1}^{m} \sum_{j=1}^{n} \pi_{i'j}^{*} \left[v(k_{i'}, h_j, x_{i'}) + u(h_j, k_{i'}, y_j) \right] \right.$$

$$+\sum_{j\in J_\pi^-} u(h_j,0,y_j)\Bigg\} >$$

$$\sum_\pi q_\pi(k_i,k_{-i},h)\left\{\sum_{i'=1}^m\sum_{j=1}^n \pi_{i'j}\left[v(k_{i'},h_j,x_{i'})+u(h_j,k_{i'},y_j)\right]\right.$$

$$\left.+\sum_{j\in J_\pi^-} u(h_j,0,y_j)\right\}$$

The second equality follows from the continuity of the functions v and u. The inequality comes from the contrary hypothesis that the sum is not upper semi-continuous. This contradicts the assumption that the matching function efficiently resolves ties. By (Reny 1999), the upper-semi-continuity of the sum of the payoffs implies that the mixed extension of the investment game is reciprocally upper semi-continuous.

The mixed extension of any game is *payoff secure* if for every array of strategies μ, and each $\varepsilon > 0$, each player i has a strategy $\overline{\mu}_i$ such that

$$\tilde{V}_i\left(\overline{\mu}_i,\mu'_{-i}\right) \geq \tilde{V}_i(\mu)-\varepsilon$$

for all μ'_{-i} in some weak neighborhood of μ_{-i}. We show that the mixed extension of the investment game is payoff secure.

First, we formalize the obvious argument that a small increase in investment won't reduce the payoff in any fixed match by much, but will must always improve partner's quality. We then show that this small increase in investment secures a payoff against any array of strategies (weakly) close to the original fixed strategies. Fix strategies μ^*_{-i} and let $v^* = \sup_k V_i\left(k,\mu^*_{-i},x_i\right)$. Let k^* be some investment level for i such that

$$\left|v^* - V_i\left(k^*,\mu^*_{-i},x_i\right)\right| < \frac{\varepsilon}{2}$$

Observe that the underlying utility function $v(\cdot,\cdot,x_i)$ is continuous, thus uniformly continuous on compact sets. Then there is a δ such that $v(k^*+\delta,h,x_i) > v(k^*,h,x_i)-\frac{\varepsilon}{2}$ for all $h\in H$. Let $\overline{R}(k,k_{-i}) = 1+\#\{j:k_j<k\}$. Since $\overline{R}(k,k_{-i})$ is the worst rank that i can have when he invests k, then $V_i\left(k^*+\delta,\mu^*_{-i},x_i\right)$ satisfies

$$\int\cdots\int\sum_\pi q_\pi(k^*+\delta,k_{-i},h)\sum_{j=1}^n \pi_{ij} v(k^*+\delta,h_j,x_i)\,d\mu^*_{-i} \geq$$

$$\int\cdots\int v\left(k^*+\delta,h_{\left(\overline{R}(k^*+\delta,k_{-i})+n-m\right)},x_i\right)d\mu^*_{-i} >$$

$$\int \cdots \int v\left(k^*, h_{\left(\overline{R}(k^*+\delta, k_{-i})+n-m\right)}, x_i\right) d\mu^*_{-i} - \frac{\varepsilon}{2} \geq$$

$$\int \cdots \int \sum_{\pi} q_\pi\left(k^*, k_{-i}, h\right) \sum_{j=1}^{n} \pi_{ij} v\left(k^*, h_j, x_i\right) d\mu^*_{-i} - \frac{\varepsilon}{2} \geq$$

$$v^* - \epsilon$$

Thus the investment $k^* + \delta$ attains a payoff that is within ϵ of the supremum of the payoffs against strategies μ^*_{-i}.

We want to show that $k^* + \delta$ attains a payoff at least $v^* - \epsilon$ against any array of strategies in some weak neighbourhood of μ^*_{-i}. Let $\mathcal{P}(H)$ and $\mathcal{P}(K)$ be the sets of probability measures on H and K respectively endowed with the weak topology. Let $\left\{\mu^t_{-i}\right\}$ be a sequence in $\mathcal{P}(H)^n \times \mathcal{P}(K)^{m-1}$that converges to μ^*_{-i} in the product topology. We want to show that for any such sequence

$$\liminf \int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*+\delta, k_{-i})+n-m\right)}, x_i\right) d\mu^t_{-i} \geq$$

$$v^* - \epsilon \tag{1.3}$$

The result then follows from the fact that the expression on the left hand side of (1.3) is a lower bound for the payoff associated with the investment $k^* + \delta$.

The order statistics $h_{(\cdot)}$ are all continuous functions, and v is continuous. Since H is compact, v is a uniformly continuous function of each worker's investment. Then for any *worker* j, if μ^t_j converges weakly to μ^*_j then

$$\lim \int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*+\delta, k_{-i})+n-m\right)}, x_i\right) d\mu^*_{-ij} d\mu^t_j =$$

$$\int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*+\delta, k_{-i})+n-m\right)}, x_i\right) d\mu^*_{-i}$$

by standard properties of weak convergence.

For any *firm* j define $\overline{R}_{-j}\left(k_i, k_{-i}\right) = 1 + \#\left\{i' \neq j : k_i > k_{i'}\right\}$. This is the worst rank that i could have if j invests more than he does. Let μ_j be an arbitrary mixed strategy for firm j. Write the left hand side of the last expression as

$$\overline{\mu}_j\left(k^*\right) \int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*, k_{-ij})+1\right)}, x_i\right) d\mu^*_{-ij}\left(k\right) +$$

$$\left[1 - \overline{\mu}_j\left(k^*\right)\right] \int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*, k_{-ij})\right)}, x_i\right) d\mu^*_{-ij}\left(k\right) \tag{1.4}$$

where $\overline{\mu}_j(k^*)$ is the probability that firm j chooses an investment strictly less than k^*. The function $\overline{\mu}_j$ describes the probability weight assigned to an open set. By standard properties of weak convergence (Billingsley 1999), $\liminf \overline{\mu}_j^t(X) \geq \overline{\mu}_j^*(X)$ for each open X, and so

$$\liminf_{t\to\infty} \left\{ \overline{\mu}_j^t(k_i) \int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*,k_{-ij})+1\right)}, x_i\right) d\mu_{-ij}^*(k) + \right.$$
$$\left. \left[1 - \overline{\mu}_j^t(k_i)\right] \int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*,k_{-ij})\right)}, x_i\right) d\mu_{-ij}^*(k) \right\} \geq$$
$$\left\{ \overline{\mu}_j^*(k_i) \int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*,k_{-ij})+1\right)}, x_i\right) d\mu_{-ij}^*(k) + \right.$$
$$\left. \left[1 - \overline{\mu}_j^*(k_i)\right] \int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*,k_{-ij})\right)}, x_i\right) d\mu_{-ij}^*(k) \right\} =$$
$$\int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*+\delta,k_{-i})+n-m\right)}, x_i\right) d\mu_{-i}^*$$

because of the fact that $h_{\left(\overline{R}(k^*,k_{-ij})+1\right)} \geq h_{\left(\overline{R}(k^*,k_{-ij})\right)}$. Combining these last two results gives that

$$\liminf \int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*+\delta,k_{-i})+n-m\right)}, x_i\right) d\mu_{-ij}^* d\mu_j^t \geq$$
$$v^* - \epsilon$$

for each $j \neq i$ and every sequence μ_j^t that converges weakly to μ_j^* which is equivalent to

$$\liminf \int \cdots \int v\left(k^* + \delta, h_{\left(\overline{R}(k^*+\delta,k_{-i})+n-m\right)}, x_i\right) d\mu_{-i}^t \geq$$
$$v^* - \epsilon$$

for each sequence μ_{-i}^t that converges to μ_{-i}^* in the product topology. Since the expression on the left hand side cannot exceed $V_i\left(k^* + \delta, \mu_{-i}^t, x_i\right)$, the investment $k^* + \delta$ can be used to secure the payoff $v^* - \epsilon$. □

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