Appendix B.
to ‘Internet Auctions with Many Traders.
(The Appendix contains the proofs of Lemma 5, Corollary 1, and Lemmas 3-9 in the proof of Theorem 2.)

**Proof of Lemma 5:** Since buyer $i$’s deviation to $\sigma$ does not change the lowest standing bid at the terminal stage, she cannot get a higher payoff after this deviation unless in state $\Gamma$ she either holds more than one high bid equal to or greater than $v_\Gamma$ with sellers in $S_\Gamma(v_\Gamma)$, or she holds a single such high bid and her valuation is strictly less than $v_\Gamma$. To see this, note that by Lemma 2 two aspects of the outcome for buyer $i$ are random when all buyers follow $\sigma^*$: (a) the price which $i$ pays to a seller where her high bid in $\Gamma$ is in $B_\Gamma^d$ could be either $v_\Gamma$ or $v_\Gamma + d$ (she wins all the units where she holds such high bids); (b) her high bid(s) equal to $v_\Gamma$ may be outbid in $G_\Gamma$ or not. If $i$ deviates to $\sigma$ in $G_\Gamma$ and the lowest standing bid at the terminal stage remains equal to $v_\Gamma$, she will still win all the units where her high bids at $\Gamma$ are strictly above $v_\Gamma$ at prices which are no less than the ones she would pay if she had not deviated. However, the expected outcomes of aspects (a) and (b) could be different because the probability with which another buyer submits a bid equal to $v_\Gamma$ at a seller in $S_\Gamma(v_\Gamma)$ where $i$ holds a high bid equal to or above $v_\Gamma$ could change as a result of this deviation.

So, suppose that in state $\Gamma$ $i$ holds $w_1$ high bids equal to $v_\Gamma$ with a set of sellers $W(i, v_\Gamma) \in S_\Gamma(v_\Gamma)$ and $w_2$ high bids in $B_\Gamma^d$ with a set of sellers $W(i, d) \in S_\Gamma(v_\Gamma)$ such that $w_1 + w_2 \geq 1$, and either $w_1 \geq 2$ or $w_2 \geq 1$ if $v_i \geq v_\Gamma$. The outcomes of (a) and (b) depend on whether other buyers submit bids higher than $v_\Gamma$ at sellers in $W(i, v_\Gamma)$ and $W(i, d)$. Since the continuation game $G_\Gamma$ is finite, by the one-deviation property we can restrict consideration to strategies $\sigma$ which involve a deviation from $\sigma^*$ only at one information set.

First, suppose that this single deviation involves an unsuccessful bid by buyer $i$. This deviation affects only the standing bid of a seller where this bid is placed, but affects neither the order in which the other buyers move, nor the strategy according to which they bid at other sellers. Thus, an unsuccessful bid does not affect the outcome.

Now suppose that $i$ deviates from $\sigma^*$ in $G_\Gamma$ by submitting a successful bid $b_i$ (i.e. a bid which becomes a high bid) at some information set $\mathcal{S}'$. To show that this causes $i$’s payoff to decrease, let us compare buyer $i$’s expected payoffs in two continuation games $l^c$ and $l^f$ which
start in information set $S'$, but in $l'$ $i$ makes the described deviation at $S'$, while in $l^e$ buyer $i$ passes at $S'$ as prescribed by $\sigma^*$. $i$ follows $\sigma^*$ after information set $S'$ in both $l^e$ and $l'$. We need to consider two different cases: Case(i): $b_i < v_T$; Case (ii): $b_i \geq v_T$.

**Case (i):** $b_i < v_T$. We will show that $i$ has the same expected payoffs in $l^e$ and $l'$, given her information at $S'$.

Let $S_{b_i+d}^e(S_{b_i+d}^f)$ denote the earliest information set in $l^e (l')$ in which the lowest standing bid is equal to $b_i + d$. $S_{b_i+d}^e$ and $S_{b_i+d}^f$ have the following properties. First, no bidder bids more than $b_i + d$ in $l^e$ prior to $S_{b_i+d}^e$ and in $l'$ prior to $S_{b_i+d}^f$, so any high bid of at least $b_i + d$ at $\Gamma$ remains such in both $S_{b_i+d}^e$ and $S_{b_i+d}^f$. Hence, in both $S_{b_i+d}^e$ and $S_{b_i+d}^f$, the sellers not in $S_1^0(b_i)$ have the same high bids and high bidders, as well as the same standing bids.

Second, $i$’s deviating high bid $b_i$ is outbid for sure prior to information set $S_{b_i+d}^f$. So, the sets of high bids held by $i$ are the same in both $S_{b_i+d}^e$ and $S_{b_i+d}^f$.

Next, let $P_{b_i+d}^e(\cdot)$ and $P_{b_i+d}^f(\cdot)$ be the probability distribution over the set of high bidders with sellers $S_1^0(b_i)$ at information set $S_{b_i+d}^e (S_{b_i+d}^f)$ given the information at $S'$ and the buyers’ strategies in the continuation game. That is, for any set $B^w$ of $\#S_1^0(b_i)$ buyers, $P_{b_i+d}^k(B^w)$ ($k \in \{e, f\}$) is the probability that at $S_{b_i+d}^k$ the set of high bidders with sellers $S_1^0(b_i)$ is exactly $B^w$.

Observe that $P_{b_i+d}^e(\cdot)$ and $P_{b_i+d}^f(\cdot)$ are identical. Specifically, both $P_{b_i+d}^e(\cdot)$ and $P_{b_i+d}^f(\cdot)$ put probability 1 on the set of high bidders which includes the subset of buyers in $D_1^e(b_i)$ whose indices are between $\#S_1^0(b_i - d) + 2$-st and $\#S_1^0(b_i) + 1$-st among the buyers in $D_1^e(b_i)$ (these buyers have the first opportunity to bid in $l^e$ and $l'$ only when the lowest standing bid reaches $b_i$). Also, $P_{b_i+d}^e(\cdot)$ and $P_{b_i+d}^f(\cdot)$ put equal probability on any subset of size $\#S_1^0(b_i - d)$ from the set of $\#S_1^0(b_i - d) + 1$ buyers in $D_1^e(b_i)$ with the lowest indices. That is, exactly one buyer from the set of $\#S_1^0(b_i - d) + 1$ buyers in $D_1^e(b_i)$ with the lowest indices is not a high bidder at either $S_{b_i+d}^e$ or $S_{b_i+d}^f$, and each of these buyers is not a high bidder with the same probability. This buyer has an opportunity to bid in $S_{b_i+d}^e (S_{b_i+d}^f)$. This is due to the fact the buyers enter the game and submit their bids if they are outbid in order of their indices and that, according to strategy $\sigma^*$, they randomize with equal probability over the relevant sets of sellers.

Since all buyers follow $\sigma^*$ in $l^e (l')$, buyer $i$’s expected payoff in $l^e (l')$ depends only on the profile of the standing bids in $S_{b_i+d}^e (S_{b_i+d}^f)$ and the probability distribution $P_{b_i+d}^e(\cdot)$ ($P_{b_i+d}^f(\cdot)$). But the profiles of the standing bids at $S_{b_i+d}^e$ and $S_{b_i+d}^f$ are identical by definition,
and we have just shown that \( P^e_{b_i+d}(\cdot) \) and \( P^f_{b_i+d}(\cdot) \) are also identical. Therefore, \( i \) has the same expected payoffs in \( l^e \) and \( l^f \).

**Case (ii):** \( b_i \geq v_\Gamma \).

If \( v_i \leq v_\Gamma \), then buyer \( i \)'s payoff is (weakly) decreasing in the number of units that she wins. If \( v_i > v_\Gamma \), then buyer \( i \)'s payoff is decreasing in the number of units that she wins in excess of 1, and she wins at least one unit for sure in each of \( l^e \) and \( l^f \), because she follows strategy \( \sigma^* \) after information set \( S' \) and the lowest standing bid at the terminal stage is equal to \( v_\Gamma \). If \( w_2 \geq 1 \), i.e. buyer \( i \) holds high bids in \( B^i \) with sellers \( W(i, d) \in S_\Gamma(v_\Gamma) \), then she wins all these units and pays either \( v_\Gamma \) or \( v_\Gamma + d \) for each of them. Also, since the lowest standing bid at the terminal stage is \( v_\Gamma \), in both \( l^e \) and \( l^f \) buyer \( i \) wins all units from sellers not in \( S_\Gamma(v_\Gamma) \) with whom she holds high bids at \( \Gamma \). The standing bids at these sellers do not change in \( l^e \).

Therefore, no matter what \( i \)'s valuation is, it is sufficient to show that both the expected number of units bought by \( i \) and the expected number of units that she buys from sellers \( S_\Gamma(v_\Gamma) \) at price \( v_\Gamma + d \) are weakly greater in \( l^f \) than in \( l^e \).

If \( i \) places her deviating bid \( b_i \) at a seller \( z \notin S^0_\Gamma(v_\Gamma) \) (i.e. \( z \)'s high bid is strictly above \( v_\Gamma \)), then \( b_i \) will never be outbid and \( i \) would for sure win \( z \)'s unit because all buyers follow \( \sigma^* \) in \( l^f \) and the lowest standing bid never exceeds \( v_\Gamma \). The expected number of units that \( i \) wins at sellers \( S_\Gamma(v_\Gamma) \) will either remain the same (if the valuation of buyer \( j_z \) who is outbid at \( z \) does not exceed \( v_\Gamma \) and so \( j_z \) will not bid in \( l^f \)), or will decrease by at most 1 (if \( j_z \)'s valuation exceeds \( v_\Gamma \) and so \( j_z \) will bid in \( l^f \)). In either case, the expected number of units that \( i \) buys from sellers \( W(i, d) \) at price \( v_\Gamma + d \) is at least as large in \( l^f \) as in \( l^e \).

In the rest of the proof, suppose that bid \( b_i \) is placed with a seller in \( S^0_\Gamma(v_\Gamma) \). Recall that the lowest standing bid reaches \( v_\Gamma \) at information set \( S^k \) in \( l^k \) \((k \in \{e, f\}) \). Let \( h(k) \) be the subset of the set of buyers \( D^k_\Gamma(v_\Gamma) \) who at information set \( S^k \) are high bidders with sellers in \( S^k_\Gamma(v_\Gamma) \).

Since all buyers follow \( \sigma^* \) in \( l^e \) \((l^f) \), buyer \( i \)'s expected payoff from trading with sellers \( S_\Gamma(v_\Gamma) \) in either game is determined by \( \#h(e) \) \((\#h(f)) \), \( w_1 \), \( w_2 \), and the value of the deviating bid \( b_i \) in \( l^f \). In particular, the expected number of units which \( i \) buys from sellers \( S_\Gamma(v_\Gamma) \) in \( l^e \) is equal to \( \pi(v_\Gamma, e) \equiv w_2 + w_1 \frac{\#S^0_\Gamma(v_\Gamma) - \#D^0_\Gamma(v_\Gamma)}{\#S^0_\Gamma(v_\Gamma) - \#h(e)} \). To understand this expression, note that \( i \) ends up buying all \( w_2 \) units from sellers in \( W(i, d) \). Also, all buyers in \( D^0_\Gamma(v_\Gamma) \) end up trading
with sellers from $S^0_i(v_T) \subset S_T(v_T)$ (by definition, $\#D^0_T(v_T) \leq \#D^0_T(v_T)$), and at information set $S^e$, $\#S^0_i(v_T) - \#h(e)$ of these buyers are not high bidders. So, the buyers in $D^0_T(v_T) \setminus h(e)$ continue to submit bids $v_T + d$ in $G_T$ after information set $S^e$ randomizing uniformly over the sellers with standing bid $v_T$. Hence, only $\#S^0_i(v_T) - \#D^0_T(v_T)$ of high bids equal to $v_T$ with sellers in $S^0_i(v_T)$ at information set $S^e$ will survive, each with equal probability, i.e. each of $i$’s $w_1$ high bids of $v_T$ will survive with probability $\frac{\#S^0_i(v_T) - \#D^0_T(v_T)}{\#S^0_i(v_T) - \#h(e)}$.

Next, note that the price at a seller in $W(i, d)$ rises to $v_T + d$ if an active buyer from $D^0_T(v_T) \setminus h(e)$ happens to place her bid $v_T + d$ at this seller. So, by a similar computation, the expected number of units that bidder $i$ buys from sellers $W(i, d) \subset S(v_T)$ at price $v_T + d$, rather than $v_T$, is equal to $\pi(d, e) = w_2 \frac{\#D^0_T(v_T) - \#h(e)+1}{\#S^0_i(v_T) - \#h(e)+1}$.

In the continuation game $l'$ we can perform similar computations taking into account $i$’s additional high bid $b_i$. Particularly, if $b_i = v_T$ ($b_i > v_T$), then the expected number of units that $i$ buys from sellers in $S_T(v_T)$ is equal to $\pi'(v_T, l') \equiv w_2 + (w_1 + 1) \frac{\#S^0_i(v_T) - \#D^0_T(v_T)}{\#S^0_i(v_T) - \#h(f)}$, while the expected number of units that she buys from sellers in $S_T(v_T)$ at price $v_T + d$, rather than $v_T$, is equal to $\pi'(d, f) = w_2 \frac{\#D^0_T(v_T) - \#h(f)+1}{\#S^0_i(v_T) - \#h(f)+1}$.

If $i$ places her deviating bid $b_i$ after the lowest standing bid has reached $v_T$ i.e. $S'$ precedes $S^e \equiv S'$, then $h(e) \equiv h(f)$ and so $\pi(v_T, l') < \pi'(v_T, l') \leq \pi''(v_T, l')$ and $\pi(d, e) < \pi'(d, f) \leq \pi''(d, f)$. Hence, $i$’s expected payoff is strictly lower in $l'$ than in $l'$.

Next, suppose that at information set $S'$ where $i$ makes her deviating bid $b_i$, the lowest standing bid is strictly below $v_T$. In this case, we will compare buyer $i$’s expectations $E\pi(v_T, e)$, $E\pi(d, e)$, $E\pi'(v_T, f)$, $E\pi''(v_T, f)$, $E\pi'(d, f)$, and $E\pi''(d, f)$ at information set $S'$. These expectations do not depend on whether $b_i = v_T$ or $b_i > v_T$, as the other buyers do not observe the value of $b_i$, and so their actions prior to information set $S'$ are independent of this value, and in either case $b_i$ remains a high bid at $S'$. Therefore, since $\pi'(v_T, f) \leq \pi''(v_T, f)$ and $\pi'(d, f) \leq \pi''(d, f)$ for each $w_1$, $w_2$ and $\#h(f)$, it is sufficient to provide the proof for $b_i = v_T$ only.

To compare $E\pi(v_T, e)$ with $E\pi'(v_T, f)$, and $E\pi(d, e)$ with $E\pi'(d, f)$, we need to characterize the probability distributions of $\#h(e)$ and $\#h(f)$ in $S^e$ and $S'$ respectively. To do so, first note that the set of high bidders with sellers in $S^0_i(v_T) \setminus S^0_i(v_T - d)$ is the same at information sets $S^e$ and $S'$ as it remains unchanged since $\Gamma$.
Let us now describe $P^e_{\Gamma_T}(\cdot)$ and $P^I_{\Gamma_T}(\cdot)$ - the probability distributions over the sets of high bidders with sellers $S^0_T(v_T)$ at information sets $S^e$ and $S^I$ respectively. $P^e_{\Gamma_T}(\cdot)$ puts probability 1 on the subset of buyers in $D^e_T(v_T - d)$ whose indices are between $\#S^0_T(v_T - 2d) + 2$-th and $\#S^0_T(v_T - d) + 1$-th among the buyers in $D^e_T(v_T - d)$ (each of these buyers has her first opportunity to bid in $l^e$ and $l^I$ only when the lowest standing bid reaches $v_T - d$), and puts an equal probability on any subset consisting of $\#S^0_T(b_i - 2d)$ buyers from the set of $\#S^0_T(b_i - 2d) + 1$ buyers in $D^e_T(b_i - d)$ with the lowest indices.

On the other hand, $P^I_{\Gamma_T}(\cdot)$ puts probability 1 on the subset of buyers in $D^I_T(v_T - d)$ whose indices are between $\#S^0_I(v_T - d) + 2$-th and $\#S^0_I(v_T)$-th among the buyers in $D^I_T(v_T - d)$ (these buyers have the first opportunity to bid in $l^e$ and $l^I$ only when the lowest standing bid reaches $v_T - d$), and puts equal probability on any subset consisting of $\#S^0_I(b_i - d)$ buyers from the set of $\#S^0_I(b_i - d) + 1$ buyers in $D^I_T(b_i - d)$ with the lowest indices. Additionally, $i$'s deviating bid $b_i = v_T$ remains a high bid in $S^I$ and is held at a seller in $S^0_I(v_T - d)$ (otherwise $b_i$ could not have become a high bid). Note that $i$ is not in $D^e_T(v_T - d)$ since at $\Gamma$ she holds at least one high bid equal to $v_T$.

To summarize, the only difference between $P^e_{\Gamma_T}(\cdot)$ and $P^I_{\Gamma_T}(\cdot)$ is that $P^e_{\Gamma_T}(\cdot)$ puts probability 1 on the buyer with the $\#S^0_I(v_T) + 1$-th index in the set $D^e_T(v_T - d)$, whom we will denote by $j'$ (this buyer is the last to place a high bid in $l^e$ prior to $S^e$) and puts zero probability on buyer $i$, while the opposite is true for $P^I_{\Gamma_T}(\cdot)$: $P^I_{\Gamma_T}(\cdot)$ puts zero probability on $j'$ who never gets to bid in $l^I$ prior to $S^I$, and puts probability 1 on buyer $i$. $P^e_{\Gamma_T}(\cdot)$ and $P^I_{\Gamma_T}(\cdot)$ put equal probabilities on other sets of bidders. That is, let $B^w_{i-j'}$ be a set of $\#S^e_T(v_T) - i$ traders that does not include buyers $i$ or $j'$. Then $P^e(B^w_{i-j'}, j') = P^I(B^w_{i-j'}, i)$.

Then, to compare $E\pi(v_T, e)$ with $E\pi'(v_T, f)$, and $E\pi(d, e)$ with $E\pi'(d, f)$, fix $B^w_{i-j'}$ - the set of traders other than $i$ and $j'$ who are high bidders with sellers in $S^0_T(v_T)$. Let us compute the conditional expectations $E\pi(v_T, e|B^w_{i-j'}, j')$, $E\pi(d, e|B^w_{i-j'}, j')$, $E\pi'(v_T, f|B^w_{i-j'}, i)$, $E\pi'(d, f|B^w_{i-j'}, i)$. For any $B^w_{i-j'}$, $\#h(f) = \#h(e)$ if $j'$'s valuation is equal to $v_T$, and $\#h(f) = \#h(e) + 1$ if $j'$'s valuation is greater than $v_T$. Inspecting the relevant formulae, it is easy to see that in each case $E\pi(v_T, e|B^w_{i-j'}, j') \leq E\pi'(v_T, f|B^w_{i-j'}, i)$, and $E\pi(d, e|B^w_{i-j'}, j') \leq E\pi'(d, f|B^w_{i-j'}, i)$ for all $B^w_{i-j'}$. Since $P^e(B^w_{i-j'}, j') = P^I(B^w_{i-j'}, i)$, we conclude that $E\pi(v_T, e) \leq E\pi'(v_T, f)$ and $E\pi(d, e) \leq E\pi'(d, f)$.

**Proof of Corollary 1:** Consider the set $N_2$ of sellers who post reserve prices equal to $v_m$. A
seller from \(N_2\) trades only if a buyer from \(M_3\) bids with her. After accounting for sellers from \(N_1\) who trade for sure, the number of sellers from \(M_3\) who are available to bid with sellers from \(N_2\) is at least \(m_3 - n_1\). This gives the lower bound on the number of sellers from \(N_2\) who trade.

To obtain the upper bound, note that on the equilibrium path buyers from \(M_2\) will bid only with sellers from \(N_1\) while the standing bids at these sellers are below \(v_m\). Consider the first time \(t\) when the lowest standing bid in the market reaches \(v_m\). With a positive probability, the realizations of random order of bidding and the randomization by buyers between the sellers among whom they are indifferent is such that at time \(t\) \(m'\) buyers from \(M_2\) are the high bidders at sellers from \(N_1\), where \(m'\) is between \(\max\{0, n_1 - m_3\}\) and \(\min\{m_2, n_1\}\). Also, with a positive probability all buyers from \(M_3\) who at time \(t\) are not winners yet, will bid at sellers from \(N_2\) first. The number of buyers from \(M_3\) who will bid in this way is equal to: \(m_3 - (n_1 - m')\). Substituting for \(m'\) we get the upper bound on the number of sellers from \(N_2\) who trade.

The proof establishing the lower and upper bounds on the number of buyers from \(M_2\) who trade is similar and is therefore omitted.

**Lemmas from the proof of Theorem 2.**

**Proof of Lemma 3.** By Claims 1 and 2, it is sufficient to compare the expected payoffs that seller \(j\) gets after setting her reserve price at \(p - d\) and at \(p\) when at least one of these prices is pivotal. Let us consider all such cases. Note that a seller posting a pivotal reserve price may fail to trade, if there are other sellers who post this reserve price or buyers with valuations equal to this price. Below we consider all possible scenarios.

1. If \(p\) is pivotal but \(p-d\) is not, then \(m_1 + n_1 < m - 1 \leq m_1 + n_1 + m_2 + n'_2\). So, irrespective of seller \(j\)'s reserve price, \(v_m\) and hence the trading price are equal to \(p\). Consequently, if \(j\) sets reserve price \(p-d\) she trades at price \(p\) for sure. If she sets reserve price \(p\), she may fail to trade if \(n_1 + n'_2 \geq m_3\) (or equivalently \(m_1 + n_1 + m_2 + n'_2 \geq m\)).

2. If \(p-d\) is pivotal, but \(p\) is not, then \(m_1 + n_1 \geq m\). So, irrespective of seller \(j\)'s reserve price, \(v_m\) (and hence the trading price) is equal to \(p-d\). Consequently, if \(j\) sets reserve price \(p\) she fails to trade. If she sets reserve price \(p-d\), she may fail to trade. The upper bound on her probability of trading will be derived below.
3. If both \( p - d \) and \( p \) are pivotal, then \( m_1 + n_1 = m - 1 \). So, if seller \( j \) posts reserve price \( p - d \) she will trade at this price for sure. If the seller sets reserve price \( p \), the trading price will be equal to \( p \) but seller \( j \) may fail to trade if \( m_2 + n_2 > 0 \).

To summarize, seller \( j \) gets a higher payoff by setting reserve price \( p - d \) rather than \( p \) if and only if the following inequality holds:

\[
(p - c) \times P(p \text{ is pivotal, } p - d \text{ is not pivotal, seller posting } p \text{ fails to trade } | c_j ) + \\
(p - d - c) \times (P(p \text{ is not pivotal, } p - d \text{ is pivotal, seller posting } p - d \text{ trades } | c_j ) + \\
P(p \text{ is pivotal, } p - d \text{ is pivotal } | c_j ) \\
\geq (p - c) \times P(p \text{ is pivotal, } p - d \text{ is pivotal, seller posting } p \text{ trades } | c_j )
\]  
(6)

Obviously, (3) implies (6), and the two are equivalent when \( c = p - d \). \( Q.E.D. \)

**Proof of Lemma 7.** Recall that both \( p \) and \( p - d \) are pivotal iff \( m_1 + n_1 = m - 1 \). Let us compute the upper bound on the probability that seller \( j \) who has cost \( c_j \) and posts price \( p \), trades at this price. Since all buyers follow strategy \( \sigma^* \), seller \( j \) can trade only with one of \( m_3 \) buyers whose valuations are strictly greater than \( p \). By corollary 1, \( n_1 \) sellers who post reserve prices below \( p \) trade for sure. Some of these \( n_1 \) sellers may trade with \( m_2 \) buyers whose valuations are equal to \( p \). Therefore, the number of buyers who can trade with sellers posting \( p \) is at most \( m_3 + \min\{0, m_2 - n_1\} \), and is strictly smaller if some buyers with valuations equal to \( p \) do not trade and some buyers with valuations above \( p \) trade with sellers whose reserve prices are below \( p \).

Seller \( j \) competes with the other \( n'_2 \) sellers who post reserve price \( p \). According to strategy \( \sigma^* \), a buyer who chooses among such sellers randomizes between them with equal probability. It follows that, given the array of buyers’ valuations and sellers’ costs, the probability that seller \( j \) trades is at most \( \min\{1, \frac{m_3 + \min\{0, m_2 - n_1\}}{n'_2 + 1}\} \).

Finally, if \( m_1 + n_1 = m - 1 \), then \( m_3 + \min\{0, m_2 - n_1\} = 1 \) if \( m_2 \leq n_1 \) and \( 0 \) if \( m_2 > n_1 \). \( Q.E.D. \)

**Proof of Lemma 8.** Recall that price \( p \) is pivotal, and \( p - d \) is not pivotal iff \( m_1 + n_1 < m - 1 \leq m_1 + n_1 + m_2 + n'_2 \).

When \( n'_2 + 1 \) sellers (including seller \( j \)) post price \( p \), and at most \( m_3 + \min\{0, m_2 - n_1\} \) buyers are available to trade with these sellers, then the lower bound on the probability that seller \( j \) posting price \( p \) fails to trade is equal to \( \max\{0, \frac{n'_2 + 1 - m_3 - \min\{0, m_2 - n_1\}}{n'_2 + 1}\} \). \( Q.E.D. \)
**Proof of Lemma 9.** First, let us focus on the right-hand side of (4). We have:

\[
E\left(\frac{1}{n_2' + 1}\right| m_1 + n_1 = m - 1, m_2 \leq n_1, c_j)P(m_1 + n_1 = m - 1, m_2 \leq n_1|c_j) = \\
\sum_{\hat{m}_1 = \max\{0, m-n\}}^{m-1} E\left(\frac{1}{n_2' + 1}\right| \hat{m}_1, n_1 = m - 1 - \hat{m}_1, m_2 \leq n_1, c_j) \times \\
P(m_1 = \hat{m}_1, n_1 = m - 1 - \hat{m}_1, m_2 \leq n_1|c_j)
\]

(7)

Now consider the right-hand side of (5). We have:

\[
E\left(\max\{0, n_2' + 1 - m_3 - \min\{0, m_2 - n_1\}\}\right|m_1 + n_1 < m - 1 \leq m_1 + n_1 + m_2 + n_2', c_j) \times \\
P(m_1 + n_1 < m - 1 \leq m_1 + n_1 + m_2 + n_2', c_j) \geq \\
E\left(\max\{0, n_2' + 1 - m_3 - \min\{0, m_2 - n_1\}\}\right|m_1 + n_1 = m - 2, n_2' \geq 2, c_j) \times \\
P(m_1 + n_1 = m - 2, n_2' \geq 2|c_j) \\
\geq 1/2 \sum_{\hat{m}_1 = \max\{1, m-n+2\}}^{m-1} E\left(\frac{n_2' - 1}{n_2' + 1}\right| m_1 = \hat{m}_1 - 1, n_1 = m - \hat{m}_1 - 1, n_2' \geq 2, c_j) \\
\times P(m_1 = \hat{m}_1 - 1, n_1 = m - \hat{m}_1 - 1, n_2' \geq 2|c_j) + \\
1/2 \sum_{\hat{m}_1 = \max\{0, m-n+1\}}^{m-2} E\left(\frac{n_2' - 1}{n_2' + 1}\right| m_1 = \hat{m}_1, n_1 = m - 2 - \hat{m}_1, n_2' \geq 2, c_j) \\
\times P(m_1 = \hat{m}_1, n_1 = m - 2 - \hat{m}_1, n_2' \geq 2|c_j)
\]

(8)

The first inequality holds because both the first and the second expressions are conditional expectations of the same non-negative function, but in the first expression we condition on a strictly larger set of events \((m_1 + n_1 < n - 1 \text{ and } m_1 + n_1 + m_2 + n_2' \geq m - 1)\) than in the second expression \((m_1 + n_1 = m - 2, \text{ and } n_2' \geq 2)\). To get the second inequality, we rewrite the expectation as summation in \(m_1\) and use the factor 1/2 to eliminate double counting, and note that \(m_3 + \min\{0, m_2 - n_1\} \leq 2\), when \(m_1 + n_1 = m - 2\).

To complete the proof, we will establish that the expression on the right-hand side of (8) is greater than the expression on the left-hand side of (7).

Let \(b (b_{-i})\) denote the array of the realized valuations of all buyers’ (all buyers other than \(i\)). Similarly, let \(c (c_{-j})\) denote the array of the realized costs of all sellers (all sellers other than \(j\)). Let \(f(p) = \min_{1 \leq i \leq m, b_{-i}, c} f^i(p|b_{-i}, c)\), and \(g(p) = \min_{1 \leq j \leq n, c_{-j}, b} g^j(p|c_{-j}, b)\). By assumption, \(f(p) > 0\) and \(g(p) > 0\ \forall p \in D\).
Recall that \( m = nk \), and let \( \alpha = \max\{\frac{2k-1}{2k}, \frac{3(1-f(p))}{3-2f(p)}\} \). We will consider two cases: \( \hat{m}_1 \geq \alpha m \) and \( \hat{m}_1 < \alpha m \). Before proceeding any further, we will need the following combinatorial result.

**Lemma 10** Consider \( N \) draws from the set \( \{a, b\} \) s.t. in each draw the probability of drawing \( a \) is at most \( \overline{q} \) and the probability of drawing \( b \) is at least \( q \). Let \( n_a \) be the number of \( a \)'s drawn in \( N \) trials. Then for \( k \geq 1 \) we have:

\[
Prob(n_a = k) \leq Prob(n_a = k - 1) \left(1 - \frac{q}{\overline{q}} \right) \frac{N - k + 1}{k}
\]

**Proof.** Let \( A_{i_1, \ldots, i_{k-1}} \) \((A_{i_1, \ldots, i_k})\) be an array of length \( n \) consisting of \( a \)'s and \( b \)'s s.t. the number of \( a \)'s in the array is \( k-1 \) \((k)\) and they occupy positions \( i_1, \ldots, i_{k-1} \) \((i_1, \ldots, i_k)\). Let \( I_{k-1} (I_k) \) denote the set of all arrays of length \( N \) that contain \( k-1 \) \((k)\) \(a\)'s and \( N-k+1 \((N-k)\) \(b\)'s. Then, \( \sum_{A_{i_1, \ldots, i_k} \in I_k} Prob(A_{i_1, \ldots, i_k}) \) and \( \sum_{A_{i_1, \ldots, i_{k-1}} \in I_{k-1}} Prob(A_{i_1, \ldots, i_{k-1}}) \).

Let us call arrays \( A_{i_1, \ldots, i_{k-1}} \in I_{k-1} \) and \( A_{h_1, \ldots, h_k} \in I_k \) adjacent if \( A_{i_1, \ldots, i_{k-1}} \) and \( A_{h_1, \ldots, h_k} \) have identical elements (either \( a \) or \( b \)) in all positions except for position \( h_j \), and \( b \) \((a)\) occupies position \( h_j \) in array \( A_{i_1, \ldots, i_{k-1}} \) \((A_{i_1, \ldots, i_k})\).

For any array \( A_{i_1, \ldots, i_{k-1}} \in I_{k-1} \) there are \( N-k+1 \) arrays in \( I_k \) that are adjacent to \( A_{i_1, \ldots, i_{k-1}} \). Any such adjacent array takes the form \( A_{i_1, \ldots, i_{k-1}, h_j} \) and is obtained from \( A_{i_1, \ldots, i_{k-1}} \) by replacing \( b \) in position \( h_j \) by an \( a \). So, for any pair of adjacent arrays we have:

\[
Prob(A_{i_1, \ldots, i_{k-1}, h_j}) \leq \frac{q}{\overline{q}}Prob(A_{i_1, \ldots, i_{k-1}}).
\]

Conversely, for any array \( A_{h_1, \ldots, h_k} \in I_k \) there are \( k \) arrays in \( I_{k-1} \) that are adjacent to it. Any such adjacent array \( A_{h_1, \ldots, h_k} \) is obtained from \( A_{h_1, \ldots, h_k} \) by replacing \( a \) in some position \( h_j' \) \({} \in \{h_1, \ldots, h_k\}\) by \( b \). Obviously, the same relationship between probabilities holds i.e.

\[
Prob(A_{h_1, \ldots, h_k}) \leq \frac{q}{\overline{q}}Prob(A_{h_1, \ldots, h_k}).
\]

Consider set \( B_{k-1} \) that contains each array in \( I_{k-1} \) replicated \( N-k+1 \) times. Similarly, let \( B_k \) be the set containing each array in \( I_k \) replicated \( k \) times. Consider a bijection \( r(.) \) between the elements of \( B_{k-1} \) and \( B_k \) such that each array in \( B_{k-1} \) corresponds to exactly one adjacent array in \( B_k \), and vice versa. As shown above, the ratio of the probability that an array \( A' \) in \( B_{k-1} \) is drawn to the probability that array \( r(A') \in B_k \) is drawn is at least \( \frac{q}{\overline{q}} \).

Hence, \[ \sum_{A_{i_1, \ldots, i_k} \in I_k} Prob(A_{i_1, \ldots, i_k}) \leq \frac{q}{\overline{q}} \frac{N-k+1}{k} \sum_{A_{i_1, \ldots, i_{k-1}} \in I_{k-1}} Prob(A_{i_1, \ldots, i_{k-1}}). \]

Q.E.D.
Now we are ready to handle cases 1 and 2 separately.

**Case 1:** $\hat{m}_1 \geq \alpha m$. We will establish that $\exists m'(p)$ s.t. if $m \geq m'(p)$, then the term corresponding to $\hat{m}_1$ in the first expression after the last inequality in (8) is greater than the term corresponding to the same $\hat{m}_1$ on the right-hand side of (7), i.e.

$$E\left(\frac{n'_2 - 1}{n'_2 + 1} | m_1 = \hat{m}_1 - 1, n_1 = m - 1 - \hat{m}_1, n'_2 \geq 2, c_j \right) P(m_1 = \hat{m}_1 - 1, n_1 = m - 1 - \hat{m}_1, n'_2 \geq 2|c_j) \geq 2E\left(\frac{1}{n'_2 + 1} | m_1 = \hat{m}_1, n_1 = m - 1 - \hat{m}_1, m_2 \leq n_1|c_j \right)$$

(10)

Note that by definition $\alpha m \geq \frac{2k-1}{2k}m = m - n/2$ which is greater than $m - n + 2$ when $n \geq 4$. Also, $\alpha m > 1$ when $m$ is sufficiently large. So, each $\hat{m}_1 \geq \alpha m$ is within the range of the summation in the first expression after the last inequality in (8) and the summation on the right-hand side of (7). The proof of inequality (10) will be provided via a sequence of claims.

**Claim 1:**

Let $P(c_{-j}|b,c_j)$ denote the probability distribution over the costs of $n-1$ sellers conditional on the array of the valuations of $m$ buyers $b$, and the cost $c_j$ of the seller $j$. Let us number all $n-1$ sellers other than $j$ from 1 to $n-1$ in some arbitrary way and let $c_i$ stand for the cost of $i$’s seller for $i \in \{1, ..., n-1\}$. By properties of conditional probability we have:

$$P(c_{-j}|b,c_j) = P(c_{n-1}|b,c_j,c_1,\ldots,c_{n-2}) \times \ldots \times P(c_{i}|b,c_j,c_1,\ldots,c_{i-1}) \times \ldots \times P(c_1|b,c_j) \quad (11)$$

Note that $P(c_i = p|b,c_j,c_1,\ldots,c_{i-1}) \geq g(p)$ for all $i \in \{1, \ldots, n-1\}$ and $\forall p$. This in combination with (11) implies the following.

Let us use notation $S^{n-1}$ to denote the set of all sellers except seller $j$. Let $\mathcal{J}^{n-m+\hat{m}_1}$ denote a collection of all possible sets containing $n-m+\hat{m}_1$ sellers, and let $B^{n-m+\hat{m}_1}$ denote a typical set in this collection. Since the sellers in (11) were numbered in an arbitrary way, $\forall B^{n-m+\hat{m}_1}$ and $\forall p < p$ we have:

$$P(\#\{i| i \in B^{n-m+\hat{m}_1}, c_i = p\} = 0|b,c_j, \{c_i \geq p \text{ iff } i \in B^{n-m+\hat{m}_1}\}) \leq (1 - g(p))^{n-m+\hat{m}_1}$$

$$P(\#\{i| i \in B^{n-m+\hat{m}_1}, c_i = p\} = 1|b,c_j, \{c_i \geq p \text{ iff } i \in B^{n-m+\hat{m}_1}\}) \leq (n-m+\hat{m}_1)(1 - g(\bar{p}))(1 - g(p))^{n-m+\hat{m}_1-1}$$

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It follows that
\[
P(n'_2 < 2, n_1 = m - 1 - \hat{m}_1, m_1 = \hat{m}_1 | c_j) = P(n'_2 < 2 | m_1 = \hat{m}_1, n_1 = m - 1 - \hat{m}_1, c_j) \leq \]
\[
(1 - g(p))^{n-m+\hat{m}_1} + (n - m + \hat{m}_1)(1 - g(p))(1 - g(p))^{n-m+\hat{m}_1-1} \leq (n - m + \hat{m}_1 + 1)(1 - g)^{n-m+\hat{m}_1}
\]
where \( g = \min_{p \in D} g(p) \). Since \( n + \hat{m}_1 - m \geq m/2k, (n - m + \hat{m}_1 + 1)(1 - g)^{n-m+\hat{m}_1} \) is decreasing in \( m \) when \( m \) is sufficiently large, and converges to zero as \( m \) goes to infinity.

**Claim 2.**
\[
P(m_1 = \hat{m}_1 - 1, n_1 = m - 1 - \hat{m}_1, n'_2 = \hat{n}_2 | c_j) \leq \frac{P(m_1 = \hat{m}_1 - 1 | n_1 = m - 1 - \hat{m}_1, n'_2 = \hat{n}_2, c_j)}{P(m_1 = \hat{m}_1 | n_1 = m - 1 - \hat{m}_1, n'_2 = \hat{n}_2, c_j)} \leq \frac{f(p)}{1 - f(p)}\frac{\hat{m}_1}{m - \hat{m}_1 + 1} \leq \frac{\alpha}{1 - f(p)(1 - \alpha) + 1/m}
\]
\[
\text{The equality holds by definition. The first inequality follows from lemma 10. The second inequality holds because } \hat{m}_1 \geq \alpha m.
\]

**Claim 3.** \( \exists m'(p) \) s.t. if \( m > m'(p) \), then
\[
E(\frac{1}{n'_2 + 1} | m_1 = \hat{m}_1, n_1 = m - 1 - \hat{m}_1, c_j) = \frac{P(m_1 = \hat{m}_1, n_1 = m - 1 - \hat{m}_1, n'_2 = 2 | c_j)}{P(m_1 = \hat{m}_1, n_1 = m - 1 - \hat{m}_1, n'_2 = 2 | c_j)} \leq \frac{1}{3} P(m_1 = \hat{m}_1, n_1 = m - 1 - \hat{m}_1, n'_2 = 2 | c_j) \leq \frac{1}{3}\left( 1 - \frac{f(p)}{1 - f(p)} + \frac{\alpha}{1 - f(p)(1 - \alpha) + 1/m} \right) \leq \frac{1}{3} P(m_1 = \hat{m}_1, n_1 = m - 1 - \hat{m}_1, n'_2 = 2 | c_j) \leq \frac{1}{6} P(m_1 = \hat{m}_1 - 1, n_1 = m - 1 - \hat{m}_1, n'_2 = 2 | c_j)
\]
\[
\text{The first inequality follows by computation. The second inequality follows from Claim 1. The third inequality follows from Claim 2.}
\]
To obtain the final inequality, note that \( \exists m(p) \) s.t. if \( m > m(p) \), then
\[
\frac{1}{1 - (n-m+\hat{m}_1+1)(1-g)^{n-m+\hat{m}_1}} < 1/6, \text{ and so the expression in brackets in less than } 1/2. \text{ Also, since } \alpha > \frac{3(1-f(p)/4)^{1/m}}{3-2(1-f(p)/4)}, \text{ we have } \frac{(1-\alpha)+1/m}{\alpha} \leq \frac{f(p)}{2(1-f(p))}, \text{ if } m > \frac{2}{1-\alpha}. \text{ Therefore, the final inequality in } (13) \text{ holds if } m \geq m'(p) \equiv \max\{\hat{m}(p), \frac{2}{1-\alpha}\}.
\]

**Claim 4.** The left-hand side of (10) is greater than \( \frac{1}{6} P(m_1 = \hat{m}_1 - 1, n_1 = m - 1 - \hat{m}_1, n'_2 = 2 | c_j) \). This is obvious.
Thus, (10) holds when $m \geq \bar{m}$.

**Case 2.** $\hat{m}_1 < \alpha m$. We will demonstrate that in this case $\exists m''(p)$ s.t. if $m \geq m''(p)$, then the very last expression in (8) is greater than the expression on the right-hand side of (7). The desired inequality can be rewritten as follows:

$$
\frac{1}{2} \sum_{\hat{m}_1 = \max\{0, m-n\}}^{\min\{\alpha m, m-3\}} \sum_{\hat{n}_2 = 0}^{n_2 + \hat{m}_1} P(m_1 = \hat{m}_1 + 1, n_1 = m - \hat{m}_1 - 3, n_2' = \hat{n}_2 + 2|c_j) \geq \min\{\alpha m, m-1\} \sum_{\hat{m}_1 = \max\{0, m-n\}}^{n_2 + \hat{m}_1} \sum_{\hat{n}_2 = 0}^{n_2 + \hat{m}_1} \frac{1}{\hat{n}_2 + 3} P(m_1 = \hat{m}_1 + 1, n_1 = m - \hat{m}_1 - 3, n_2' = \hat{n}_2 + 2|c_j) \geq
$$

$$
\left(\frac{g(p)}{1-g(p)}\right)^2 \frac{f(p) + \ldots + f(p-d)}{1-f(p) - \ldots - f(p)} \frac{(m - \hat{m}_1 - 2)(m - \hat{m}_1 - 1)(m - \hat{m}_1)}{(\hat{n}_2 + 2)(\hat{n}_2 + 1)(\hat{m}_1 + 1)} \times P(m_1 = \hat{m}_1, n_1 = m - \hat{m}_1 - 1, n_2' = \hat{n}_2|c_j)
$$

Hence,

$$
\frac{\hat{n}_2 + 1}{\hat{n}_2 + 3} P(m_1 = \hat{m}_1 + 1, n_1 = m - \hat{m}_1 - 3, n_2' = \hat{n}_2 + 2|c_j) \geq \left(\frac{g(p)}{1-g(p)}\right)^2 \frac{f(p) + \ldots + f(p-d)}{1-f(p) - \ldots - f(p)} \frac{(m - \hat{m}_1 - 2)(m - \hat{m}_1 - 1)(m - \hat{m}_1)}{(\hat{n}_2 + 3)(\hat{n}_2 + 2)(\hat{m}_1 + 1)} \times P(m_1 = \hat{m}_1, n_1 = m - \hat{m}_1 - 1, n_2' = \hat{n}_2|c_j)
$$

Let us now establish that $\frac{(m-\hat{m}_1-2)(m-\hat{m}_1-1)(m-\hat{m}_1)(\hat{n}_2+1)}{(n_2+3)(n_2+2)(\hat{m}_1+1)}$ goes to infinity as $m$ increases. First $\frac{\hat{n}_2+1}{n_2+3} \geq 1/3$. Second, $\frac{m-\hat{m}_1}{n_1+1} \geq \frac{1-\alpha}{\alpha+1/m} \geq \frac{1-\alpha}{1+\alpha}$. Third, $\frac{m-\hat{m}_1-1}{n_2+2} \geq \frac{(1-\alpha)m-1}{\alpha+2}$ which is greater than $\frac{(1-\alpha)}{2(1/k+1-\alpha)}$ when $m \geq \frac{2}{1-\alpha}$.

Finally, $m - \hat{m}_1 - 2 \geq (1-\alpha)m - 2 \geq m \times \min\{\frac{1}{2k}, \frac{f(p)}{3-2}(\hat{p})\} - 2$, which converges to infinity as $m$ increases. It follows that $\exists m''(p)$ s.t. if $m \geq m''(p)$, then (14) holds. So, the Lemma holds if we choose $N \geq \max\{m'(p), m''(p)\}$.

Q.E.D.